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Relativistic Time for Terrestrial Circumnavigations

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Relative to a clock at rest on the Earth's surface, the time recorded by an ideal clock after a circumnavigation of the Earth depends not only on the speed and altitude but also on the direction of the circumnavigation and on the rotational speed of the Earth. Such a clock may run either fast or slow, depending on the direction and ground speed for the circumnavigation. This directional dependence should be perceptible with commercial jet speeds and cesium beam clocks.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

In a recent letter in *Nature*,[1] I briefly developed a prediction of the theory of relativity that states that the relativistic time offset accumulated by a clock during circumnavigation of the Earth depends both on the direction of the circumnavigation and on the Earth's rotational speed. This rather straightforward prediction of the theory seems to have been largely overlooked in the past. Moreover the magnitude of the expected time offset is enhanced by the Earth's surface speed to the point where it should be perceptible with ordinary international jet speeds and modern atomic clocks. Consequently, I suggested an experimental test of this prediction of a directional dependence. The purpose of this paper is to present a more detailed description of these relativistic effects and to discuss more fully their implications.

Our objective, therefore, is to predict the outcome of the following idealized, but in principle executable experiment. Two similar clocks initially are located together at rest on the Earth's surface at the equator. They are carefully tested and intercompared to assure that they keep the same time, i.e., that they record the same number of "ticks" over long time intervals. Then one of the clocks is placed in a jet airplane that rapidly climbs to its cruising altitude, flies completely around the Earth in the equatorial plane at this cruising altitude and with a constant ground speed with no stops, and then rapidly descends to the departure point where the other clock remained at "rest." The question to be considered is the following: If the two clocks read the same time before the flight, what will be their relative time readings after the flight? It will be assumed the clocks record proper time.

Notice that this question does not ask about Doppler shifts or other instantaneous effects between the two clocks. No signals are transmitted between the clocks during the flight.

II. THEORY

Almost all calculations in relativity begin with an assumed prior knowledge of the space–time metric. It will be a sufficiently accurate approximation to assume that the Earth is a spherically symmetric source of scalar gravitational potential χ, and that the Earth rotates on its axis once a day in an otherwise flat space. The choice of a metric for a *nonrotating* space greatly simplifies the calculation; metrics for rotating spaces always involve cross terms between the infinitesimal coordinate space and time intervals, and in this case time synchronization for the hypothetical coordinate clocks is intransitive, which induces a certain arbitrariness in the meaning of time.[2] The nonrotating metric for the region outside a spherically symmetric source of gravitational potential is the Schwarzschild metric[3]:

$$ds^2=\left(1+\frac{2\chi}{c^2}\right)c^2dt^2 - \left[\frac{dr^2}{1+(2\chi/c^2)}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right]. \quad (1)$$

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In this expression the coordinate time interval dt and the corresponding coordinate space intervals dr, $rd\theta$, and $r\sin\theta d\phi$ are the coordinate intervals between two infinitesimally close space–time events, such as two close points in the path of a physical clock. Since the proper time interval $d\tau$, recorded by such a moving clock during corresponding coordinate time and space intervals, is ds/c, Eq. (1) gives

$$d\tau=\left[\left(1+\frac{2\chi}{c^2}\right)-\left(\frac{u_r^2}{1+(2\chi/c^2)}+u_\theta^2+u_\phi^2\right)\frac{1}{c^2}\right]^{1/2}dt, \quad (2)$$

where

$$u^2=u_r^2+u_\theta^2+u_\phi^2=(dr^2/dt^2)+r^2[(d\theta^2/dt^2)+(\sin^2\theta d\phi^2/dt^2)],$$

which is the square of the coordinate speed. At this point it is convenient to reduce Eq. (2) for a slow speed ($u^2\ll c^2$) and weak field ($|\chi|\ll c^2$) approximation:

$$d\tau\cong[1+(\chi/c^2)-(u^2/2c^2)]dt, \quad (3)$$

where terms of order higher than u^2/c^2 and χ/c^2 have been neglected.

If a clock follows a certain definite path in space, P, beginning at a point A at the coordinate time t_A and ending at a point B at the coordinate time t_B, the finite proper time interval recorded by that clock is given by the line integral of $d\tau$ along the path P:

$$\Delta\tau=\tau_B-\tau_A=\int_{t_A}^{t_B}P\left(1+\frac{\chi[r(t)]}{c^2}-\frac{u^2(t)}{2c^2}\right)dt. \quad (4)$$

The implication here is that the scalar potential χ and the coordinate speed u are definite known quantities at each instant of coordinate time t along the path P. Now let us apply these results to the case in question, viz., clocks circumnavigating the Earth at the equator.

Figure 1 suggests a view of the Earth as perceived by a nonrotating observer looking down on the North pole from a great distance. The "stationary" reference clock at rest on the surface follows a circular path as the Earth turns on its axis. Let the radius of the Earth be R and its angular speed be Ω. Then the coordinate speed for this clock is $u_0=R\Omega$, and the scalar potential is $\chi_0=-GM/R$, where G is the gravitational constant and M is the Earth's mass. Application of Eq. (4) gives the proper time interval recorded by the stationary clock:

$$\Delta\tau_0=\int_{t_{A_0}}^{t_{B_0}}\left(1-\frac{GM}{c^2R}-\frac{R^2\Omega^2}{2c^2}\right)dt=[1-(GM/c^2R)-(R^2\Omega^2/2c^2)]\Delta t_0, \quad (5)$$

where Δt_0 is the corresponding coordinate time interval.

A similar calculation gives the proper time recorded by the flying clock. Let us neglect the small differences in recorded times during the brief ascent to and descent from the cruising altitude. Assume the flying clock follows a circular path around the Earth at an altitude h above the surface and with a constant ground speed v, which is positive for eastward and negative for westward motion (see Fig. 1). The

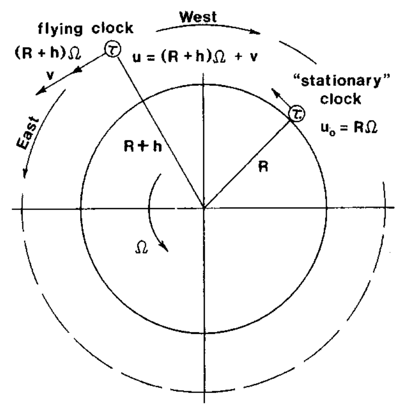

Fig. 1. Coordinate speeds in a nonrotating space for a clock at rest on the Earth's surface at the equator and an airborne clock circumnavigating the Earth in the equatorial plane.

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relativistic law for the addition of velocities gives the correct coordinate speed for the flying clock:

$$u=\frac{(R+h)\Omega+v}{1+(R+h)\Omega v/c^2}.$$

The denominator in this case contributes only to higher orders that have already been neglected, so the approximation $u=(R+h)\Omega+v$ is sufficient. Hence application of Eq. (4) to the flying clock gives:

$$\Delta\tau=\int_{t_A}^{t_B}\left(1-\frac{GM}{c^2R(1+h/R)}-\frac{[R\Omega(1+h/R)+v]^2}{2c^2}\right)dt$$

$$=\left(1-\frac{GM}{c^2R(1+h/R)}-\frac{[R\Omega(1+h/R)+v]^2}{2c^2}\right)\Delta t, \quad (6)$$

where the scalar potential in this case is

$$\chi=-GM/(R+h)$$

and the corresponding coordinate time interval is Δt.

The paths that the two clocks follow in this experiment are different, but since they both begin and end at the same spacetime points, the coordinate time intervals for both are the same, i.e., $\Delta t=\Delta t_0$. Hence division of Eq. (6) by Eq. (5) gives the ratio of the recorded proper time intervals, independently of the common coordinate time interval:

$$\frac{\Delta\tau}{\Delta\tau_0}=\frac{1-\dfrac{GM}{c^2R(1+h/R)}-\dfrac{[R\Omega(1+h/R)+v]^2}{2c^2}}{1-\dfrac{GM}{c^2R}-\dfrac{R^2\Omega^2}{2c^2}}. \quad (7)$$

This expression, which can be simplified considerably, gives the desired relationship between the recorded times. A little algebraic manipulation, assuming $h\ll R$ and retaining only lowest order terms, produces the result stated in Ref. 1:

$$\Delta\tau/\Delta\tau_0=1+(gh/c^2)-(2R\Omega v+v^2)/2c^2, \quad (8)$$

where $g=GM/R^2-R\Omega^2$, the measured surface value of the acceleration of gravity at the equator.

Let us define a quantity δ to be the difference in the times recorded by the flying and reference clocks divided by the time of the reference clock:

$$\delta=(\Delta\tau-\Delta\tau_0)/\Delta\tau_0.$$

A rearrangement of Eq. (8) gives the predicted value for δ:

$$\delta_{th}=(gh/c^2)-(2R\Omega+v)v/2c^2. \quad (9)$$

Let us call δ the time offset between the two clocks. (It may also be appropriate to regard δ as the difference in rates for the clocks, but the notion of rate has too many preconceived and misleading connotations.[4]) Notice that the offset defined here is a unique (invariant) quantity that is quite independent of the frame of reference of an observer who measures it. Moreover, δ is a negative quantity if the flying clock "runs" slow, while it is positive if the flying clock "runs" fast. Equation (9) predicts that the flying clock may run fast or slow, depending on the direction of the circumnavigation (the sign of v).

Cutler has derived a more general relation for the expected offset for clocks that circumnavigate the Earth at any latitude,[5] not just at the equator. If λ is the latitude ($\lambda=0$ at the equator), Eq. (9) becomes:

$$\delta_{th}=(gh/c^2)-[(2R\Omega\cos\lambda+v)v/2c^2].$$

Hence the directional dependence of the offset is reduced at higher latitudes in proportion to $\cos\lambda$.

III. DISCUSSION AND CONCLUSIONS

The effect of altitude [h in Eq. (9)] on the time-keeping behavior of terrestrial clocks is associated with the gravitational red shift and is rather well understood. It predicts that a clock in a stronger gravitational field (h small) records

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less time than a similar clock in a weaker gravitational field (h large).

A quite remarkable feature of Eq. (9), however, is the prediction of a directional dependence on the ground speed. Let us consider for illustration a few examples with $h=0$, a case that would be difficult to achieve in practice but that tends to simplify our thought experiments. Suppose the circumnavigation is westward with a ground speed $v=-R\Omega$. In this case the flying clock "stands still" in the nonrotating reference space while the clock on the ground traverses a circular path in one day. The predicted offset is:

$$\delta_{th}=R^2\Omega^2/2c^2,$$

with the ground clock running slow, which means the flying clock would appear to run fast.

Now suppose the circumnavigation is still westward but with a ground speed $v=-2R\Omega$. The predicted offset is zero, which upon reflection is not surprising because in this case the two clocks move with the same coordinate speed in the same circle but in opposite directions.

As a final example, suppose the flying clock circumnavigates the Earth in an eastward direction. In this case v is positive and the flying clock always has a coordinate speed that is greater than the clock on the ground. Consequently the flying clock runs slow, regardless of the magnitude of the ground speed.

At high ground speeds, where $|v|\gg R\Omega$ so that $\delta_{th}\cong gh/c^2-v^2/2c^2$, the directional dependence becomes imperceptible and the flying clock runs slow for either direction. This case would apply to a clock in an Earth satellite with a low orbit ($h\ll R$).[6]

Probably the most interesting result of this study is the prediction that the offsets that can be expected with typical commercial international jet speeds are about an order of magnitude greater than the inherent rms fluctuations between cesium beam clocks.[7] The offsets are measurable with ordinary jet speeds because the relatively large surface speed of the Earth ($2R\Omega=930$ m/sec$\cong$2000 mph) plays a dominant role. Typical international jet flights go at an altitude $h=10$ km$\cong$33 000 ft and a ground speed $v=300$ m/sec$\cong$ 670 mph. With these values Eq. (9) predicts offsets of $\delta(\text{west})=+2.1\times10^{-12}$ for a westward circumnavigation and $\delta(\text{east})=-1.0\times10^{-12}$ for an eastward circumnavigation. Since the inherent rms fluctuations[7] for portable cesium beam clocks $\delta(\text{noise})\cong10^{-13}$, which is an order of magnitude less than the predicted offsets, the expected signal to noise ratio is favorable and an experimental test of this directional dependence seems quite feasible. At the same time such an experiment would test the conventional interpretation of the theory, which predicts that macroscopic physical clocks run slow when moving relative to a nonrotating space. In other words such a test would provide an empirical resolution of the famous clock paradox with macroscopic physical clocks.

The assumption here, as is common in relativity theory, has been that all macroscopic physical clocks, irrespective of design or complexity, record proper time (assuming of course that their time keeping is not influenced by drift, damage, misalignment, failure, or other impairment of the constituent mechanisms). This hypothesis of the theory,[8] which seems quite firmly based ultimately on the principle of relativity and the principle of equivalence, nevertheless has yet to be established by observation. That it can be so established is now within our grasp.

APPENDIX

An anonymous referee has suggested that I point out that the east–west directional dependence described here is a purely special relativistic effect, in spite of the apparent use of the general theory. Although this statement is true insofar as the directional dependence alone is concerned, it is certainly not true to suggest that the general theory is not required to predict offsets for terrestrial circumnavigations. Since differences in the gravitational field are involved, the special theory is inadequate.

It is interesting, nevertheless, to derive a predicted offset based solely on the special theory. The basis for proper time in this case is the Minkowski metric, which applies in a nonrotating (inertial) reference system. The infinitesimal proper time interval, analogous to Eq. (3) of the text, is given by

$$d\tau=[1-(u^2/2c^2)]dt. \qquad \text{(A.1)}$$

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To draw an analogy as close as possible to the terrestrial situation, one might imagine circumnavigation by clocks of a thin rotating spherical shell of negligible mass, of radius R and angular speed Ω, and located in interstellar (flat) space where $\chi=$constant. Following a derivation identical to that of the main text, but employing Eq. (A.1) for infinitesimal proper time intervals, the predicted offset to the same order of approximation becomes

$$\delta_{sr}=-[2R\Omega^2h+(2R\Omega+v)v]/2c^2. \qquad \text{(A.2)}$$

Hence the directional dependence follows from a purely special relativistic argument, but of course it would be quite incorrect to use this expression to predict offsets for terrestrial circumnavigations (with $h\neq0$). Although it is similar to Eq. (A.2), Eq. (9) includes the effect of the gravitational red shift, which results from a general relativistic metric and would be nonnegligible in an actual experiment. In fact an experiment employing macroscopic clocks, similar to the one suggested here, would not only test the clock hypothesis (clocks record proper time), but would also test the equivalence principle of general relativity through its dependence on altitude. Therefore, to state that the effect is purely special relativistic could be misleading.

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