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# New Geometric Representation of the Lorentz Transformation

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Most geometric representations of the Lorentz transformation use either imaginary angles or imaginary space-time coordinates. A new and simple representation is proposed using only real coordinates and real angles, and allowing a graphical illustration of the major features of the Lorentz transformation.

EVER since Poincare and Minkowski introduced the four-dimensional space-time formalism, numerous graphical representations of the Lorentz transformation have been proposed which illustrate the physical meaning of the transformation. They are exposed in most treatises on relativity, in particular, in those of Becker, Born, and Möller. The illustrative merit of this representation is somehow weakened by one or more of the following features: imaginary time (or space) coordinates; imaginary angle of rotation; units of length (and of time) changing from an axis to another and determined by the intersection of the axes with a family of hyperbolas. We propose here a geometric representation which uses only real coordinates, real angles, and where the units of length (and of time) are the same in both systems. The two sets of coordinates of an event appear as covariant and contravariant variables. This feature which is not essential to the graphical representation can be brought up as a useful background for the study of general relativity.

We start out with a brief mathematical introduction.

## MATHEMATICAL INTRODUCTION

Consider the two axes $ou'$, $ox'$ such that angle $(x'ou')=\psi$ of Fig. 1. The coordinate lines $x'=\text{const}$, $u'=\text{const}$ are parallel to the coordinate axes and the coordinates $(u',x')$ of a point $P$ are obtained by drawing from $P$ parallels to $ox'$, $ou'$. The slope $s'$ of a line $AH$ joining two points $A$, $H$ is equal to the ratio $Q'H/AQ'$ as shown in Fig. 1, $s'=\bar{Q}'\bar{H}/\bar{A}\bar{Q}$.

Drawing $ox$ perpendicular to $ou'$ and $ou$ perpendicular to $ox'$ we have a new set of oblique axes with an angle $(xou)=\pi-\psi$. The coordinates $u$, $x$ of a point $P$ with respect to the new axes are obtained in the same manner as for $u'x'$ and they are shown in Fig. 1. Again the slope of line $AH$ is given by

$$s=\bar{Q}\bar{H}/\bar{A}\bar{Q}.$$

The relationship between the two sets of coordinates is given by

$$x=\frac{1}{\sin\psi}(x'+u'\cos\psi),$$
$$u=\frac{1}{\sin\psi}(x'\cos\psi+u'). \tag{1a}$$

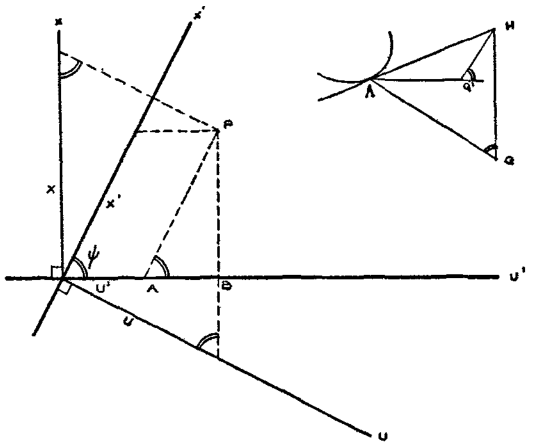

FIG. 1. Graphical representation of four-dimensional, space-time coordinate system.

The inverse transformation is

$$x' = \frac{1}{\sin\psi}(x - u\cos\psi),$$
$$u' = \frac{1}{\sin\psi}(-x\cos\psi + u). \tag{1b}$$

The proof of these transformations is straightforward and we shall derive only the second one as an example. Figure 1 shows that $\bar{O}\bar{B} = \bar{O}\bar{A} + \bar{O}\bar{B} = u' + x'\cos\psi$ and that

$$u = \bar{O}\bar{C} = \bar{O}\bar{B}/\sin\psi,$$

and this establishes the first Eq. (1a). One may remark that Eqs. (1a) and (1b) can be deduced from each other by interchanging the primes and changing $\cos\psi$ to $(-\cos\psi)$, i.e., $\psi$ to $(\pi - \psi)$.

The relationship between the values of the slope of a curve (at a given point) can be obtained by differentiating Eqs. (1a) and finding their ratios. The result is

$$\frac{dx}{du} = \frac{\dfrac{dx'}{du'} + \cos\psi}{1 + \cos\psi \dfrac{dx}{du'}}. \tag{2}$$

One can also derive this relation geometrically using for a curve tangent to $AH$ at $A$

$$\frac{dx'}{du'} = \frac{Q'H}{AQ'}; \quad \frac{dx}{du} = \frac{QH}{AQ}.$$

Upon setting $(x,u) \equiv (x^1x^2)$ and $(x'u') = (x^1x^2)$ one recognizes that this is the simplest example of covariant and contravariant coordinates i.e.,

$$x^i = g^{ij}x_j \quad x_i = g_{ij}x^j.$$

The metric tensors $g^{ij}$, $g_{ij}$ are given in Eqs. (1a) (1b).

## APPLICATION TO THE LORENTZ TRANSFORMATION

In special relativity the space-time coordinates $(x,t)$ of an event in an inertial system are related to the coordinates $(x',t')$ of the same event in another inertial system by the so-called Lorentz transformation. The second system $S'$ is moving with a constant velocity $v$ with respect to the first system $S$, $v$ being directed in the common axis direction $x$ (or $x'$). If one sets $u = ct$, $u' = ct'$, $\beta = v/c$ and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$ the Lorentz transformation can be written $y = y'$, $z = z'$ (if these axes are, respectively, parallel)

$$x = \gamma(x' + \beta u),$$
$$u = \gamma(\beta u' + u). \tag{3a}$$

The inverse transformation is then

$$x' = \gamma(x - \beta u),$$
$$u' = \gamma(-\beta x + u). \tag{3b}$$

With the time coordinates $u$, $u'$, $\beta$ is the relative velocity of the two systems and all velocities will be measured in $c$-units, the velocity of light in vacuum being one. The second postulate of special relativity requires all velocities be less than one. One verifies easily that relations (3a,b) are identical with relations (1a,b) when one sets

$$v/c = \beta = \cos\psi \text{ and therefore, } \gamma = 1/\sin\psi. \tag{4}$$

With this interpretation of Eqs. (1a,b) one can then proceed to study graphically the simple but important consequences of the Lorentz transformation.

The rigorous measurement of the length of a rod in a given system consists in recording the positions *at the same instant* of the ends of the rod. Thus let a rod $AB$ of length $l'$ lying along the $x'$-axis be at rest in $S'$ (Fig. 2). The world lines of the ends are, respectively, $AA'$ and $BB'$. In order to record their simultaneous positions one draws a parallel to the $x$-axis in $S$ and a parallel to the $x'$-axis in $S'$. The length is found to be $A'B'=l'$ in $S'$ and $A''B''=l$ in $S$, $l$ being less than $l'$ as Fig. 2 shows. Exactly:

$$1/\gamma=\sin\psi=l/l' \quad \text{or} \quad l=l'(1-\beta^2)^{\frac{1}{2}}.$$

Thus a moving rod appears shorter to a stationary observer. Consider now a clock fixed in $S'$ and sending two light-flashes at times $u_C'$, $u_D'$. The world line of the clock is $CD$ and according to $S$ events $C$, $D$ occurred at times $u_C$, $u_D$. Figure 2 readily shows that

$$1/\gamma=\sin\psi=\frac{u_D'-u_C'}{u_D-u_C},$$

or

$$u_D'-u_C'=(u_D-u_C)(1-\beta^2)^{\frac{1}{2}}.$$

A moving clock runs slower if compared to that of a stationary observer. These relationships are of course entirely reciprocal.

Figure 2 shows also that events $A'B'$ simultaneous in $S'$ occur at different times in $S$; $C$, $D$ coincident in $S'$ occur at different locations in $S$.

The velocity transformation can be derived analytically or graphically. Equation (2) can be

$$\dot{x}=\frac{\dot{x}'+\beta}{1+\beta\dot{x}'},$$

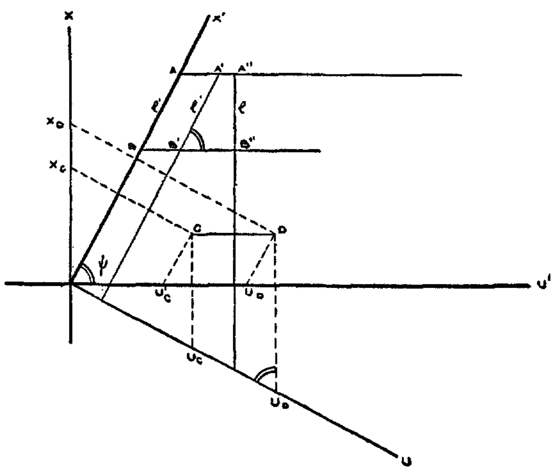

FIG. 2. Lorentz transformation between inertial reference systems.

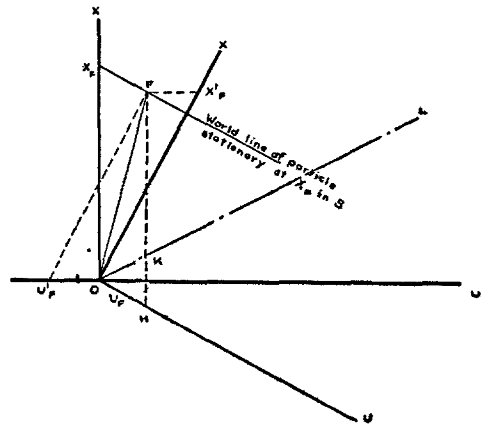


FIG. 3. Example of a particle traveling at a speed in excess of the speed of light.

written

$$\dot{x}=\frac{\dot{x}'+\beta}{1+\beta\dot{x}'},$$

where $\dot{x}=dx/cdt$, and $\dot{x}'=dx'/cdt'=dx'/du'$.

The propagation of a flash of light starting at the origin at time $u=u'=0$ is described by $OL$, and one can easily show that the slope of $OL$ is unity in both systems. The world line of any material particle must have at all times a slope less than one. The world line of the origin of system $S'$ is $ou'$.

Assume that a particle leaves the common origin of $S$, $S'$ at time $u=u'=0$ with a speed larger than that of light. The particle collides with a particle stationary at $S$ at point $x_F$. From our assumption the slope of the world line $OF$ is larger than that of $OL$, i.e.,

$$\frac{HF}{OH}>\frac{HK}{OH}=1.$$

Then the collision $F$ would take place at a *negative* time $u_F'<0$, i.e., the collision would occur before the particle has left the origin (Fig. 3)! This example shows how essential is the second postulate of special relativity.