Differing observations on the landing of the rod into the slot

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In the usual rod and slot paradox a rod falls into a slot due to gravity. Many thought experiments have been conducted where the presence of gravity is eliminated with the rod and slot approaching each other along a line joining their centers. In these experiments the line of motion is not parallel to the axis of the rod or the slot. We consider the cases for which the rod falls into the slot and the rod does not fall into the slot, each from the perspective of the co-moving frames of the rod and the slot. We show that if the rod falls into the slot as determined by Galilean kinematics, the same conclusion is valid for relativistic kinematics. Our conclusion emphasizes that the passing (or crashing) of the rod is unaffected by relativistic kinematics. This determination does not depend on the magnitude of the velocity, but only on the proper lengths and the proper angles of the rod and slot with the line of motion.
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I. INTRODUCTION

The Lorentz equations describe the relation between space-time event coordinates (x, y, z, t) and (x', y', z', t') of the same event from two inertial frames assuming the equivalence of inertial frames and constancy of the speed of light.[1] The Lorentz transformation describes the differences in the observation depending on the velocity of the observer. If the length of an object at rest is L_0, then when it moves relative to a coordinate system with speed v, it is contracted from L_0 to L according to the relation:

$$L = L_0/\gamma, \tag{1}$$

where $\gamma = 1/(\sqrt{1-v^2/c^2})$ and c is the speed of light in vacuum. In a co-moving frame with the object, the object's length will be measured as L_0. Because of the symmetrical nature of the Lorentz transformations, the two objects in relative motion will observe each other as contracted from their respective co-moving frames.

Over the years the Lorentz transformation has given rise to a number of paradoxes. A common one-dimensional version is the pole in the barn problem[2] which focuses on the question of simultaneity. Later versions include the rod and hole paradox, first discussed in Ref. 3 and later in Ref. 4.

In the rod and slot paradox we consider a rod of length L_R sliding along a solid surface that has a slot of length L_S. The length of the rod is L_R/γ according to an observer co-moving with the slot and the length of the slot is L_S/γ according to an observer co-moving with the rod. Thus, for $L_R = L_S$, an observer co-moving with the slot observes that the length of the rod is shorter than the length of the slot, and predicts that the rod will fall into the slot. However, an observer co-moving with the rod observes that the length of the slot is shorter than the length of the rod and predicts that the rod will not fall into the slot. The paradox is defined by the contradictory answers obtained by the co-moving observers (with the rod and the slot) to two questions: Which is shorter, the rod or the slot? Does the rod fall into the slot or not?

Although Rindler[3] advanced the idea of differing notions of rigidity to explain the observations from the frames of the rod and the slot, Shaw[4] wrote that as the rod starts to "fall," it experiences accelerated motion. Under those conditions it will look curved from one frame and straight from another. Shaw[4] further wrote that gravity and accelerated motion can be removed from the paradox by considering a system where there is no gravity and the analysis can be done entirely using the special theory of relativity. He considered a slot and a rod approaching each other (as observed from an inertial frame S). The rod has a large velocity along the x-axis with respect to S, and the slot has a small velocity in the z-axis (with respect to S) moving such that their centers coincide at $t=0$ in frame S. He showed that while an observer in the slot observes that the rod fell into it with their lengths aligned, an observer on the rod observes that it went into the slot at an angle and thus could go into the slot even though the rod is longer than the slot.

Grøn and Johannesen[5] animated the fall of the rod into the slot by a computer program that transforms the coordinates from one frame to another by the Lorentz transformation and displays the view from the co-moving frames of the rod and the slot. The graphics vividly show that the shape and inclination of the rod are very different as observed from the two inertial frames. The animations confirm Rindler's idea that the "rigidity of the rod" can vary as observed by the two frames. The question of whether the nature of a physical effect depends on the frame of reference from which the effect is observed is addressed in Ref. 5 by emphasizing the need for a relativistic theory of elasticity. Grøn and Johannesen[5] note that although a break in the rod is a physical effect, the bending of the rod is not, because observers in different reference frames observe the extent of this bending differently.

Marx[6] observed that as the velocity of the rod increases, the observation from the co-moving frame of the slot would be that the rod rotates and contracts, but the line of motion of the end points of the rod remains the same. This observation indicates that the rod's passing or not passing through the slot is unaffected by the magnitude of the approach velocity. Martins[7] has emphasized that any one-to-one space-time coordinate transformation, even if different than Lorentz, does not lead to a contradiction; a point-to-point collision between objects as observed in one frame transforms to a collision in the other frame, even if the space-time coordinates are different in the two frames for the collision. Thus the passing of the rod or otherwise is consistent in both the frames, even if the temporal sequences of certain events are relative.

The rod and slot paradox was recently revisited in Ref. 8. According to Ref. 8 the car and hole paradox described in Ref. 9 and the rod and hole paradox described in Refs. 3 and 4 do not give rise to a contradiction when explained on the basis of stress propagation; that is, the difference in observed speed of stress propagation compensates exactly for the differences in the observed length.

In this paper we present a variant of the rod and slot paradox with motion in two directions but only with constant velocity. Our contribution is to show how the non-invariance of the angles and proper lengths comes into play. In this scenario the line of motion, that is, the line joining the centers of the rod and slot, is not aligned with either the axis of the rod or the slot, there is no gravity, and thus there is no stress or propagation of stress. We show that whether the rod passes through the slot (or not) is determined only by the proper length of the rod, the proper length of the slot, the proper angle between the axis of the rod and the line of motion, and the proper angle between the axis of the slot and the line of motion. This determination is independent of the approach velocity and relativistic kinematics and is the same for Galilean kinematics.

II. THE ROD AND THE SLOT APPROACHING UNDER NO GRAVITY

Imagine a metal sheet located somewhere in space where there is no gravity. Let the sheet be on the x-y plane and contain a slot of length L. Also consider a rod of length L traveling in such a way so that it passes through the slot from one side of the x-y plane to the other side. For this purpose it has a velocity component along the length of the rod, say the x-axis, and another along the negative z-axis. This situation is similar to a plane landing on a runway.

Given our assumption of no gravity we can eliminate the effects of stress, stiffness, and propagation of stress. We take the center of the slot as the origin of the system and make the following observations from the rod's reference frame (see Fig. 1):

- The initial condition is such that the center of the rod is at $x=-a$, $z=+b$.
- The y coordinate is not relevant because all events occur on the x-z plane along the axis of the slot.
- The velocity v_x of the approaching slot along the x-axis is large and comparable to c.
- The velocity v_z of the approaching slot in the z direction is small.
- To land perfectly into the slot we assume that $a/v_x=b/v_z$. This assumption will make the rod land exactly into the slot after a time $t=a/v_x=b/v_z$.
- It is possible to make both b and v_z go to zero while maintaining $a/v_x=b/v_z$, making the system reduce to the conventional rod and slot paradox, with the rod sliding parallel to the length of the slot. As shown in the following, our analysis offers a solution to the general case, but a solution to this limiting case still depends on the value of the ratio b/v_z $(=a/v_x)$.

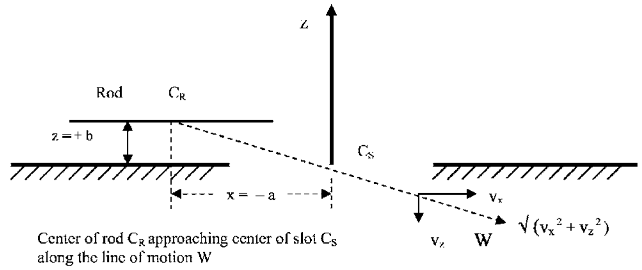

Fig. 1. Initial conditions of the rod and slot. Observations from the co-moving frame of the rod.

III. THE OBSERVATIONS OF THE ROD AND THE SLOT

The axis of neither the rod nor the slot is collinear with the line of the relative velocity W. We use the following notation (see Figs. 2–5) to designate the relevant angles:

Φ: Acute angle between the axis of the rod and W (the line of motion) as observed from the co-moving frame F of the rod.

θ: Acute angle between the axis of the rod and W as observed from the co-moving frame S of the slot.

α: Acute angle between the axis of the slot and W as observed from the co-moving frame S of the slot.

β: Acute angle between the axis of the slot and W as observed from the co-moving frame F of the rod.

Φ and α are the proper angles in the respective co-moving frames. From the Lorentz contraction these angles appear larger from the other frame (unless they are zero or 90°, in which case the two angles will appear the same from the other frame). Thus

$$\theta > \Phi \quad \text{and} \quad \beta > \alpha. \tag{2}$$

The relation between the angles is

$$\tan\theta = \gamma \tan\Phi, \tag{3a}$$

$$\tan\beta = \gamma \tan\alpha, \tag{3b}$$

where

$$\gamma = \frac{1}{\sqrt{1-(v_x^2+v_z^2)/c^2}}. \tag{4}$$

In the following we consider three cases.

Case I: $\theta=\alpha$. An observer co-moving with the slot observes the rod landing with both their axes aligned. In this

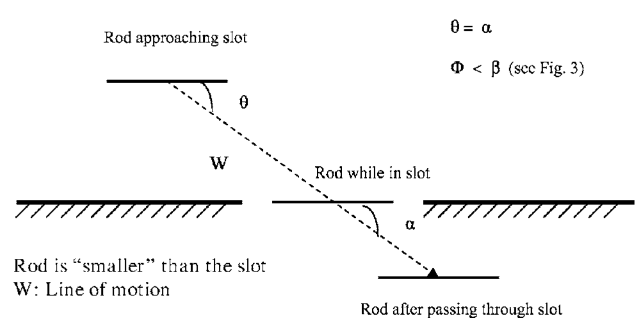

Fig. 2. The rod went through the slot. Observations from the co-moving frame of the slot.

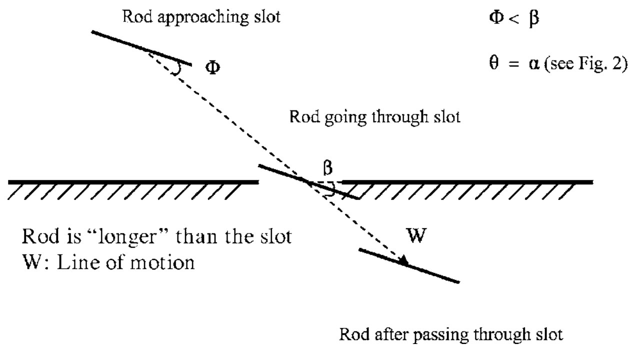

Fig. 3. The rod went through the slot. Observations from the co-moving frame of the rod.

case (see Fig. 2) the rod goes into the slot with its axes aligned and the rod is smaller than the slot (all observations are from frame S).

An observer co-moving with the rod (frame F) observes this same case differently. Consider the relation between Φ and β when $\theta=\alpha$. If we incorporate the inequalities of Eq. (2) with this equality, we obtain $\Phi<\theta=\alpha<\beta$; thus we have $\Phi<\beta$ when $\theta=\alpha$. In other words, when an observer in frame S observes an aligned landing, an observer in frame F observes a landing with the leading edge of the rod tilted toward the slot (see Fig. 3). We call this alignment a favorable alignment because it facilitates the rod passing through the slot even though the rod is longer than the slot. Figures 2 and 3 depict the observations of frames F and S in case I when the rod does pass through the slot.

Case II: $\Phi=\beta$. Observers in frame F co-moving with the rod observe that the rod is aligned with the slot. In this case an observer in frame S will observe (Fig. 4) that the rod is approaching with its leading edge tilted upward, $\theta>\alpha$. The rod does not go into the slot (see Fig. 5), because according to frame F, it is longer than the slot. But according to frame S (see Fig. 4), the rod, even though smaller, is unfavorably aligned. Figures 4 and 5 depict the observations of the frames F and S in case II when the rod does not go through the slot, that is, it crashes. Thus the two frames disagree on the alignments of the rod and the slot as well as on their lengths in both cases. The rod passing through the slot is a reality in case I; the rod not passing through the slot is a reality in case II. Observers in frames F and S agree on these realities in both cases. Only the reasons for the occurrences are different: either assigned to inequality in lengths or favorable/unfavorable alignments.

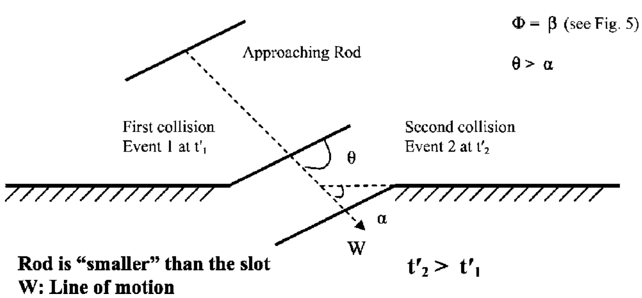

Fig. 4. The rod did not go through the slot. Observations from the co-moving frame of the slot.

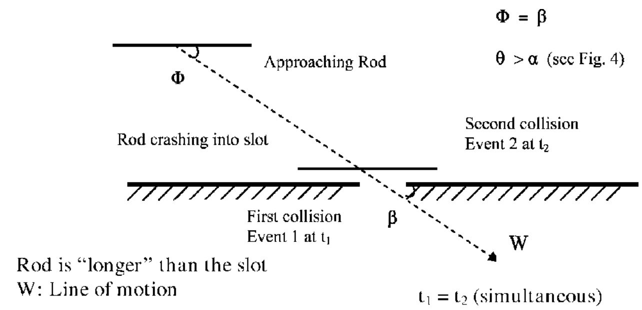

Fig. 5. The rod did not go through the slot. It crashed. Observations from the co-moving frame of the rod.

In case II the collision of both edges of the rod on to the slot is simultaneous according to the rod (Fig. 5). These two events are not simultaneous according to the slot (Fig. 4). This conclusion is intriguing as conventional thinking precludes the second collision after the first collision. The stress or disturbances from the trailing edge collision cannot travel to the leading edge before the second collision occurs. This aspect has been discussed in detail in Ref. 10.

Case III: $\Phi=\alpha$. Consider a situation where the proper angles between the axis of the rod with the line of motion and that of the slot with the line of motion are equal. This case maintains the symmetry between the two linear objects and the two corresponding frames. In this case, when we assume the proper lengths of the rod and the slot to be the same, observers in frame F (co-moving with the rod) would observe the rod entering the slot in a favorable alignment. But these observers would also observe that the length of the rod is longer than the length of the slot. The favorable alignment and the bigger length compensate for each other and the rod goes through the slot by a whisker.

Similarly, observers in frame S (co-moving with the slot) would observe the rod to be smaller and unfavorably aligned; both these aspects offset each other and the rod goes through the slot by a whisker. These results are expected because the condition $\Phi=\alpha$ maintains the symmetry between the two linear objects.

IV. GALILEAN KINEMATICS

Let L_R and L_S denote the length of the rod and the slot, respectively. When we consider the problem from the Galilean point of view, with no change in the lengths or the angles, we find that the projection of the slot parallel to the line of motion plays no role in the passing of the rod through the slot. Similarly, the projection of the rod parallel to the line of motion does not contribute to a minimum slot length requirement. Thus we find that whether the rod passes through the slot or not is determined only by the projections of the rod and slot on the line perpendicular to the line of

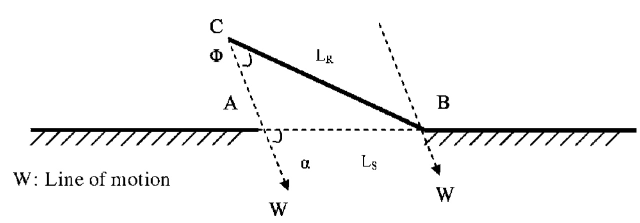

Fig. 6. Galilean kinematics: The rod just passing through the slot.

motion; that is, when the rod passes through the slot, the projection of the rod perpendicular to the line of motion passes through the projection of the slot perpendicular to the line of motion. In other words when $L_R \sin \Phi < L_S \sin \alpha$, the rod falls into the slot (see Fig. 6).

In Fig. 6 the rod lands at an angle, and when the leading edge of the rod coincides with the front edge of the slot, the rod and slot form two sides of a triangle. We assume that the rod just passes through the slot, that is, the line joining the trailing edge of the rod and the back edge of the slot coincides with the line of motion and forms the third side of the triangle. If we apply the law of sines for triangle ABC in Fig. 6, and observe that $\sin \alpha = \sin(180 - \alpha)$, we obtain the relation $L_R \sin \Phi = L_S \sin \alpha$. This relation holds for the case when the rod just manages to pass through. Thus we derive the condition that whenever $L_R \sin \Phi < L_S \sin \alpha$, the rod falls into the slot.

When the line of motion coincides with the axis of the rod, a pinhole slot of almost zero length is sufficient to let a very long rod pass through; in this case $\Phi = 0$ and $L_R \sin \Phi = 0$. Note that we have assumed that the centers of the rod and slot approach each other along the line of motion. In the conventional rod and slot paradox, the center of the rod is just a little bit above the center of the slot and gravity is required to aid the fall.

V. RELATIVISTIC KINEMATICS

For relativistic kinematics the same considerations would be valid with the proper lengths and proper angles replaced by the lengths and angles observed in an arbitrary inertial frame. From the slot's co-moving frame $L_S \sin \alpha$ is unaltered, because this frame will observe the proper length of the slot, and the angle α without any change. The length of the rod L_R and the angle Φ will be changed to a contracted length L_{RA} and angle β by the slot, but in such a way that $L_R \sin \Phi = L_{RA} \sin \beta$.

Thus we find that the condition $L_{RA} \sin \beta < L_S \sin \alpha$ is identical to the condition $L_R \sin \Phi < L_S \sin \alpha$ (where L_{RA} is the contracted length of the rod). A similar logic can be presented for the co-moving frame of the rod. In other words the condition that determines whether the rod will pass through the slot is unchanged as we switch from Galilean to relativistic kinematics in either of the inertial frames.

VI. SUMMARY

The three cases are interesting in that apart from their proper lengths, the orientation of the rod and slot with the line of motion plays a key role in determining whether the rod falls into the slot. We have shown that whether the rod passes through the slot is established by determining if $L_R \sin \Phi < L_S \sin \alpha$. This inequality is unaffected by relativistic kinematics and the magnitude of the velocity. The four (proper) quantities L_R, L_S, Φ, and α determine the result both under Galilean and relativistic kinematics. We see that for planar motion there is distortion of the angles as well as linear dimensions and the reasons attributed for physical occurrences are assigned to dimensions and to angles. The physical occurrence is the same from different inertial frames, but the observers assign different reasons for the observations.

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