# Master NPAC

# Cosmology – Lesson 2

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L. Le Guillou
(llg@lpnhe.in2p3.fr)

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## 1. Distance, metrics and curvature

In a reference frame where the coordinates are $\widetilde{\mathrm{x}} : x^\mu$, the distance $\mathrm{d}s$ between two points separated by $\mathrm{d}x^\mu$ is:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

Where $g_{\mu\nu}$ is the metric tensor.

If the space is curved, you cannot cancel all the second derivatives of $g_{\mu\nu}$ relatively to the coordinates.

The connection coefficients (Christoffel symbols) describe the effects of parallel transport in curved space:

$$\Gamma_{\nu\mu\rho} = \frac{1}{2}\left(g_{\mu\nu,\rho} - g_{\rho\mu,\nu} + g_{\nu\rho,\mu}\right) \qquad \Gamma^\nu_{\mu\rho} = \frac{1}{2}g^{\nu\sigma}\left(g_{\mu\sigma,\rho} - g_{\rho\mu,\sigma} + g_{\sigma\rho,\mu}\right) \quad \text{where} \quad g_{\mu\nu,\rho} = \partial_\rho g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\rho}$$

To take properly into account the effects of parallel transport in a curved space, the derivative should be replaced by the *covariant* derivative,

$$q^{\mu}_{;\nu} = \frac{\mathrm{D}q^{\mu}}{\mathrm{D}x^{\nu}} = \frac{\partial q^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\rho} q^{\rho} = q^{\mu}_{,\nu} + \Gamma^{\mu}_{\nu\rho} q^{\rho}$$

The curvature may be expressed through the Riemann curvature tensor:

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta} - \Gamma^{\alpha}_{\sigma\delta}\Gamma^{\sigma}_{\beta\gamma}$$

and with the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar $R^{\alpha}_{\alpha}$:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \qquad R^{\alpha}_{\alpha} = g^{\mu\nu} R_{\mu\nu}$$

In such a curved space, free particles follow the *geodesics*, defined by:

$$\frac{\mathrm{D}p^{\mu}}{\mathrm{D}\tau} = 0 \qquad \text{with} \quad p^{\mu} = m\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \qquad \frac{\mathrm{D}q^{\mu}}{\mathrm{D}s} = \frac{\mathrm{d}q^{\mu}}{\mathrm{d}s} + \Gamma^{\mu}_{\nu\rho} q^{\nu} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}s}$$

this provides the trajectories of free particles along the geodesics:

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} + \Gamma^{\mu}_{\nu\rho} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} = 0$$

## 2. General Relativity: Einstein's equation

The Einstein field equations link the curvature of spacetime, described with the Ricci curvature tensor $R_{\mu\nu}$, to the contents in matter and energy through the stress-energy tensor $T_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} R^{\alpha}_{\alpha} g_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

This may also be written using the Einstein's tensor $G_{\mu\nu}$,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R^{\alpha}_{\alpha} g_{\mu\nu}$$

And the field equation then becomes:

$$G_{\mu\nu} - \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Dimensions: $[R_{\mu\nu}] = [G_{\mu\nu}] = [\Lambda] = \mathrm{L}^{-2}$ ; $[T_{\mu\nu}] = \mathrm{ML}^{-1}\mathrm{T}^{-2}$ ; $[G] = \mathrm{M}^{-1}\mathrm{L}^{3}\mathrm{T}^{-2}$.

## 3. The Friedman-Lemaître-Robertson-Walker metric

As the universe is considered to be homogeneous and isotropic, its metric should be the most symetric solution of the Einstein equation, with a uniform curvature. This metric is the Friedman-Lemaître-Robertson-Walker (FLRW) metric:

$$\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - R^2(t) \left( \mathrm{d}\chi^2 + S_k^2(\chi) \left( \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2 \right) \right) \qquad S_k(\chi) = \begin{cases} \sin\chi & k = +1 \\ \chi & k = 0 \\ \sinh\chi & k = -1 \end{cases}$$

With the following *comoving* coordinates $\widetilde{\mathbf{x}} : x^\mu = (ct, \chi, \theta, \varphi)$.

In this equation, $R(t)$ is the universe scale factor ($[R(t)] = \mathrm{L}$). $a(t) = R(t)/R(t_0)$ is the dimensionless scale factor, with the convention $a(t_0) = 1$ at current time $t_0$.

The metric may also be written:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - R_0^2 a^2(t)\left(\mathrm{d}\chi^2 + S_k^2(\chi)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right)$$

Or, if we define $r = S_k(\chi)$,

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - R_0^2 a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right)$$

### 3.1. FLRW metric

To get a properly dimensioned metric tensor, we may also use the coordinates: $\widetilde{\mathbf{x}} : x^\mu = (x^0, x^1, x^2, x^3) = (ct, R_0 r, R_0\theta, R_0\varphi)$. The metric then becomes:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = c^2\mathrm{d}t^2 - a^2(t)\left(\frac{R_0^2\mathrm{d}r^2}{1-kr^2} + r^2\left(R_0^2\mathrm{d}\theta^2 + \sin^2\theta R_0^2\mathrm{d}\varphi^2\right)\right)$$

The metric and the inverse metric are diagonal, and the metric coefficients are:

$$\begin{aligned}
g_{00} &= 1 & g^{00} &= \frac{1}{g_{00}} = 1 \\
g_{11} &= -\frac{a^2(t)}{1-kr^2} & g^{11} &= \frac{1}{g_{11}} = -\frac{1-kr^2}{a^2(t)} \\
g_{22} &= -a^2(t)r^2 & g^{22} &= \frac{1}{g_{22}} = -\frac{1}{a^2(t)r^2} \\
g_{33} &= -a^2(t)r^2\sin^2\theta & g^{33} &= \frac{1}{g_{33}} = -\frac{1}{a^2(t)r^2\sin^2\theta}
\end{aligned}$$

With this convention, $g_{\mu\nu}$ and $g^{\mu\nu}$ are dimensionless.

### 3.2. Christoffel symbols for the FLRW metric

$$\Gamma_{\nu\mu\rho} = \frac{1}{2}\left(g_{\mu\nu,\rho} - g_{\rho\mu,\nu} + g_{\nu\rho,\mu}\right) \qquad \Gamma^\nu_{\mu\rho} = \frac{1}{2}g^{\nu\sigma}\left(g_{\mu\sigma,\rho} - g_{\rho\mu,\sigma} + g_{\sigma\rho,\mu}\right)$$

The only non-zero Christoffel symbols are:

$$\Gamma^0_{11} = \frac{1}{c}\frac{a\dot{a}}{1-kr^2} \qquad \Gamma^0_{22} = \frac{1}{c}a\dot{a}\,r^2 \qquad \Gamma^0_{33} = \frac{1}{c}a\dot{a}\,r^2\sin^2\theta$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{c}\frac{\dot{a}}{a} \qquad \Gamma^1_{11} = \frac{1}{R_0}\frac{kr}{1-kr^2} \qquad \Gamma^1_{22} = -\frac{1}{R_0}r(1-kr^2) \qquad \Gamma^1_{33} = -\frac{1}{R_0}r(1-kr^2)\sin^2\theta$$

$$\Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{c}\frac{\dot{a}}{a} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{R_0}\frac{1}{r}$$

$$\Gamma^2_{33} = -\frac{1}{R_0}\sin\theta\cos\theta \qquad \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{R_0}\frac{\cos\theta}{\sin\theta}$$

With the chosen convention, $[\Gamma^\mu_{\nu\rho}] = \mathrm{L}^{-1}$.

### 3.3. Ricci curvature tensor and Einstein's tensor

For the FLRW metric, the Ricci tensor is:

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a}$$
$$R_{11} = \frac{1}{1-kr^2}\left(\frac{\ddot{a}a}{c^2} + \frac{2\dot{a}^2}{c^2} + \frac{2k}{R_0^2}\right)$$
$$R_{22} = r^2\left(\frac{\ddot{a}a}{c^2} + \frac{2\dot{a}^2}{c^2} + \frac{2k}{R_0^2}\right)$$
$$R_{33} = r^2\sin^2\theta\left(\frac{\ddot{a}a}{c^2} + \frac{2\dot{a}^2}{c^2} + \frac{2k}{R_0^2}\right)$$

The Ricci scalar:

$$R^\alpha_\alpha = -6\left[\frac{\ddot{a}}{a}\frac{1}{c^2} + \frac{\dot{a}^2}{a^2}\frac{1}{c^2} + \frac{k}{R_0^2 a^2}\right]$$

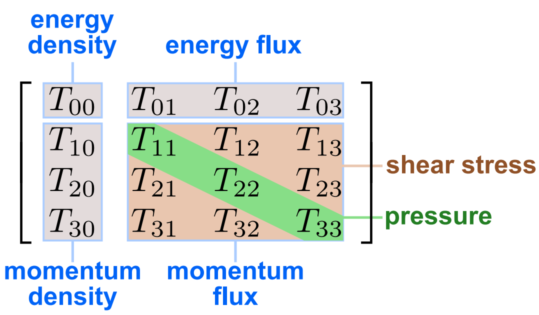
And the Einstein's tensor:

$$G_{00} = 3\frac{\dot{a}^2}{a^2}\frac{1}{c^2} + 3\frac{k}{R_0^2 a^2}$$
$$G_{ii} = g_{ii}\left[2\frac{\ddot{a}}{a}\frac{1}{c^2} + \frac{\dot{a}^2}{a^2}\frac{1}{c^2} + \frac{k}{R_0^2 a^2}\right]$$

with $i \in \{1, 2, 3\}$.

### 3.4. Stress-energy tensor

The stress-energy tensor (or energy-momentum tensor) describes the density and flux of energy and momentum in spacetime.




For a perfect fluid (no viscosity) in thermodynamic equilibrium, in a flat Minkovski spacetime, the stress-energy tensor is:

$$T_{\mu\nu} = \begin{pmatrix} \varepsilon = \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

This may be written as

$$T_{\mu\nu} = (\varepsilon + p)\frac{U_\mu U_\nu}{c^2} - p g_{\mu\nu}$$

In the comoving reference frame, the same equation gives ($\mathbf{u} = 0$):

$$T_{00} = (\varepsilon + p)\frac{U_0 U_0}{c^2} - p g_{00} = \varepsilon + p - p = \varepsilon$$

$$T_{11} = \frac{pa^2}{1 - kr^2}$$

$$T_{22} = pa^2 r^2$$

$$T_{33} = pa^2 r^2 \sin^2\theta$$

$$T_{ii} = -p g_{ii} \qquad \text{for } i \in \{1, 2, 3\}$$

## 4. The Friedmann equations

Using the previous results, we could find the Friedmann equations:

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{R_0^2 a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3c^2}\varepsilon$$

$$\frac{\ddot{a}}{a} - \frac{\Lambda c^2}{3} = -\frac{4\pi G}{3c^2}(\varepsilon + 3p)$$

Where $\varepsilon = \rho c^2$ is the energy density, and $p$ is the pressure of the fluid filling the universe. We have two equations for three unknowns: $a(t)$, $\varepsilon(t)$ and $p(t)$.

The expansion rate $H(t)$ is defined by:

$$H(t) = \frac{\dot{R}(t)}{R(t)} = \frac{\dot{a}(t)}{a(t)}$$

We may define a critical density $\varepsilon_c = \rho_c c^2$:

$$H^2(t) = \frac{8\pi G}{3}\rho_c(t) = \frac{8\pi G}{3c^2}\varepsilon_c(t)$$

$$\rho_c(t) = \frac{3}{8\pi G}H^2(t) \qquad \varepsilon_c(t) = \frac{3c^2}{8\pi G}H^2(t)$$

At present time $t_0$,

$$H_0 = H(t_0) \simeq 70\,\text{km/s/Mpc}$$
$$\rho_{c,0} = \rho_c(t_0) \simeq 9 \times 10^{-27}\,\text{kg}\,\text{m}^{-3} \simeq 1.4 \times 10^{11}\,\text{M}_\odot\,\text{Mpc}^{-3}$$
$$\varepsilon_{c,0} \simeq 5200\,\text{MeV}\,\text{m}^{-3} \simeq 5 \text{ protons per m}^3$$

Densities may be expressed as function of the critical density:

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} = \frac{\varepsilon(t)}{\varepsilon_c(t)}$$

## 5. Fluid equation and state equation

Using the hypothesis that the expansion is an adiabatic process ($\delta Q = 0$), We get:

$$\delta Q = dE + pdV = 0 \qquad \text{i.e.} \qquad dS = 0$$

from which we deduce:

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + p) = 0$$

For a given fluid, with an equation of state $p = w\varepsilon = w\rho c^2$,

$$\frac{\dot{\varepsilon}}{\varepsilon} = -3(w+1)\frac{\dot{a}}{a} \qquad \varepsilon(t) = \varepsilon(t_0)a(t)^{-3(w+1)}$$

For non-relativistic matter:

$$p_m \ll \varepsilon_m = \rho_m c^2 \qquad w_m \simeq 0 \qquad \varepsilon_m(t) = \varepsilon_m(t_0)/a(t)^3$$

For light, relativistic matter (photons, neutrinos,...):

$$p_r = \frac{\varepsilon_r}{3} \qquad w_r = \frac{1}{3} \qquad \varepsilon_r(t) = \varepsilon_r(t_0)/a(t)^4$$

For a cosmological constant $\Lambda$:

$$p_\Lambda = -\varepsilon_\Lambda \qquad w_\Lambda = -1$$

## 6. Redshift and distances

**Redshift.** The observed redshift of distant galaxies $z$ is defined as:

$$z \equiv \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} - 1 \qquad 1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{a(t_{\text{obs}})}{a(t_{\text{emit}})} = \frac{1}{a(t)}$$

**Comoving distance $\chi$ and proper distance $d_P$.** The comoving distance $\chi$ is the (dimensionless) distance between two points measured at the present cosmological time. It is constant for objects moving within the Hubble flow (with no peculiar velocity).

$$\chi = \int_{t_{\text{emit}}}^{t_0} \frac{c\mathrm{d}t'}{R_0\, a(t')}$$

The proper distance is the distance between objects at a given cosmological time,

$$d_P(t) = \int_0^\chi R(t)\, \mathrm{d}\chi = R(t)\chi = R_0 a(t)\chi$$

This is the distance we measure to nearby objects.

The variation with time of $d_P(t)$ gives:

$$v_P(t) = \dot{d}_P(t) = \dot{R}(t)\chi = \frac{\dot{R}(t)}{R(t)}R(t)\chi = \frac{\dot{a}(t)}{a(t)}d_P(t) = H(t)d_P(t)$$

**Angular distance $d_A$.** If we know the transverse size $\ell$ of a distant object (when the photons were emitted), and if we measure the apparent angular diameter $\delta\theta$ of the same object, we can get the *angular distance* $d_A$ to this object:

$$d_A = \frac{\ell}{\delta\theta} = R(t_{\text{emit}})S_k(\chi) = R_0 a(t_{\text{emit}})S_k(\chi) \qquad d_A(z) = (1+z)^{-1}R_0 S_k(\chi)$$

**Luminosity distance $d_L$.** When observing an object of known luminosity, we could define the *luminosity distance* $d_L$ in terms of the relationship between the absolute magnitude $M$ and the apparent magnitude $m$ of that object:

$$m - M = \mu = 5\log_{10}\left(\frac{d_L}{10\,\text{pc}}\right) = 5\log_{10} d_L - 5$$

where $\mu$ is the *distance modulus*. If the absolute magnitude is known (for a *standard candle*), then the luminosity distance $d_L$ can be measured.

The luminosity distance can be expressed using the metric:

$$d_L = \frac{1}{a(t_{\text{emit}})} R_0 Sk(\chi) \qquad d_L(z) = (1+z)R_0 S_k(\chi) = (1+z)^2 d_A$$

**Particle horizon distance $d_H$ (*aka* horizon distance).** The most distant objects you can see, at least in theory, are the one which emitted light at $t = 0$ and which is just reaching us on Earth now, *i.e.* at $t = t_0$. The horizon is the spherical surface of radius $d_H(t_0)$ centered on Earth, beyond which we cannot see because light coming from objects lying outside the horizon have not had time to reach us. The current horizon distance can be written:

$$d_H(t_0) = \int_0^{t_0} \frac{c\mathrm{d}t}{a(t)}$$


Figure 1: *The Einstein field equation, painted on an old steam engine, in the "Train Cemetery" close to Uyuni, Bolivia.*