UPMC SORBONNE UNIVERSITÉS

Master NPAC

Cosmology – Lesson 3

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Contents

1 The Friedmann equations

The Friedmann equations may be written:

$$\frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{3c^2}\varepsilon(t) + \frac{\Lambda c^2}{3} - \frac{kc^2}{R_0^2 a^2(t)} \tag{1}$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3c^2}\left(\varepsilon(t) + 3p(t)\right) + \frac{\Lambda c^2}{3} \tag{2}$$

Where $\varepsilon(t) = \rho(t)c^2$ is the energy density, and $p(t)$ is the pressure of the fluid filling the universe. We have two equations for three unknowns: $a(t)$, $\varepsilon(t)$ and $p(t)$.

The expansion rate $H(t)$ is defined by:

$$H(t) = \frac{\dot{R}(t)}{R(t)} = \frac{\dot{a}(t)}{a(t)}$$

We may define a critical density $\varepsilon_c = \rho_c c^2$:

$$H^2(t) = \frac{8\pi G}{3}\rho_c(t) = \frac{8\pi G}{3c^2}\varepsilon_c(t)$$

$$\rho_c(t) = \frac{3}{8\pi G}H^2(t) \qquad \varepsilon_c(t) = \frac{3c^2}{8\pi G}H^2(t)$$

At present time t_0,

$$H_0 = H(t_0) \simeq 70\,\mathrm{km/s/Mpc}$$

$$\rho_{c,0} = \rho_c(t_0) \simeq 9 \times 10^{-27}\,\mathrm{kg\,m^{-3}} \simeq 1.4 \times 10^{11}\,\mathrm{M_\odot\,Mpc^{-3}}$$

$$\varepsilon_{c,0} \simeq 5200\,\mathrm{MeV\,m^{-3}} \simeq 5 \text{ protons per m}^3$$

Densities may be expressed as function of the critical density:

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} = \frac{\varepsilon(t)}{\varepsilon_c(t)} \tag{3}$$

1.1 The fluid equation

Using the hypothesis that the expansion is an adiabatic process ($\delta Q = 0$), We get:

$$\delta Q = dE + pdV = 0 \qquad \text{i.e.} \qquad dS = 0 \tag{4}$$

from which we deduce:

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + p) = 0 \tag{5}$$

For a given fluid, with an equation of state $p = w\varepsilon = w\rho c^2$,

$$\frac{\dot{\varepsilon}}{\varepsilon} = -3(w+1)\frac{\dot{a}}{a} \qquad \varepsilon(t) = \varepsilon(t_0)a(t)^{-3(w+1)} \tag{6}$$

For non-relativistic matter:

$$p_m \ll \varepsilon_m = \rho_m c^2 \qquad w_m \simeq 0 \qquad \varepsilon_m(t) = \varepsilon_m(t_0)/a(t)^3$$

For light, relativistic matter (photons, neutrinos,. . .):

$$p_r = \frac{\varepsilon_r}{3} \qquad w_r = \frac{1}{3} \qquad \varepsilon_r(t) = \varepsilon_r(t_0)/a(t)^4$$

1.2 The cosmological constant as a fluid with negative pressure

The Friedmann equations may be rewritten as:

$$\frac{\dot{a}^2(t)}{a^2(t)} + \frac{kc^2}{R_0^2 a^2(t)} = \frac{8\pi G}{3c^2}\varepsilon(t) \tag{7}$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3c^2}\left(\varepsilon(t) + 3p(t)\right) \tag{8}$$

by absorbing the Λ term in the density and pressure terms:

$$\varepsilon \longrightarrow \varepsilon + \varepsilon_\Lambda \qquad p \longrightarrow p + p_\Lambda \qquad \text{with} \qquad \varepsilon_\Lambda = \frac{\Lambda c^2}{8\pi G} \qquad p_\Lambda = -\varepsilon_\Lambda \tag{9}$$

The cosmological constant is then interpreted as a fluid of constant density, with a negative pressure $p_\Lambda = w_\Lambda \varepsilon_\Lambda$ with $w_\Lambda = -1$. The cosmological constant may be seen as the simplest type of "*dark energy*".

1.3 Open or closed universe

If the universe contains a mixing of several component i with different equations of state $p_i = w\varepsilon_i$, the first Friedmann equation could be written:

$$H^2(t) = \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2 a^2(t)} \qquad \text{where} \quad \varepsilon = \sum_i \varepsilon_i \tag{10}$$

Using the critical density $\varepsilon_c(t)$, this can be rewritten as:

$$-\frac{kc^2}{R_0^2 a^2(t)} = H^2(t)\left[1 - \Omega(t)\right] \qquad \text{where} \quad \Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)} \tag{11}$$

As the sign of the left member of this equation cannot change, the same is true for the right term as well. It means that if $\Omega > 1$ (supercritical), then $k = +1$ and the universe is closed and positively curved, and this will stay true forever; if $\Omega < 1$ (subcritical), $k = -1$ and the universe is open and negatively curved. If Ω is exactly equal to 1, the universe is open and flat ($k = 0$).

Equation (11) is also verified at current time t_0 ($a(t_0) = 1$):

$$-\frac{kc^2}{R_0^2} = H_0^2\left[1 - \Omega_0\right] \tag{12}$$

By replacing $-kc^2/R_0^2$ by its expression, we get the Friedmann equation rewritten with the relative densities at current time t_0:

$$\begin{aligned}
\frac{H^2(t)}{H_0^2} &= \frac{\varepsilon(t)}{\varepsilon_{c,0}} + (1-\Omega_0)a^{-2} \\
\frac{H^2(t)}{H_0^2} &= \frac{\varepsilon_m(t) + \varepsilon_r(t) + \varepsilon_\Lambda(t) + \cdots}{\varepsilon_{c,0}} + (1-\Omega_0)a^{-2} \\
\frac{H^2(t)}{H_0^2} &= \frac{\varepsilon_{m,0}a^{-3} + \varepsilon_{r,0}a^{-4} + \varepsilon_{\Lambda,0} + \cdots}{\varepsilon_{c,0}} + (1-\Omega_0)a^{-2} \\
\frac{H^2(t)}{H_0^2} &= \Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} + \cdots + (1-\Omega_0)a^{-2}
\end{aligned}$$

where $\varepsilon_{c,0} = \varepsilon_c(t_0) = 3c^2 H_0^2/8\pi G$ is the critical density now (at $t = t_0$).

1.4 Age of the universe

From the previous equation we can deduce the age t_0 of an expanding universe:

$$H^2(t) = \left(\frac{\mathrm{d}a}{a\,\mathrm{d}t}\right)^2 = H_0^2\left[\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} + \cdots + (1-\Omega_0)a^{-2}\right] \tag{13}$$

Which gives:

$$\mathrm{d}t = \frac{1}{H_0}\frac{\mathrm{d}a}{\sqrt{\Omega_{m,0}a^{-1} + \Omega_{r,0}a^{-2} + \Omega_{\Lambda,0}a^2 + \cdots + (1-\Omega_0)}} \tag{14}$$

$$t_0 = \int_0^{t_0}\mathrm{d}t = \frac{1}{H_0}\int_0^{a(t_0)=1}\frac{\mathrm{d}a}{\sqrt{\Omega_{m,0}a^{-1} + \Omega_{r,0}a^{-2} + \Omega_{\Lambda,0}a^2 + \cdots + (1-\Omega_0)}} \tag{15}$$

In the most general case, this integral should be computed numerically.

1.5 Distances

Once the Friedmann equation is solved, we would know $a(t)$. We may then compute the various distances defined in lesson 2. The comoving coordinate χ to an object at redshift z is then;

$$\chi = \int_{t_{\text{emit}}}^{t_{\text{obs}}}\frac{c}{R_0}\frac{\mathrm{d}t}{a(t)} = \frac{c}{R_0}\int_{a_{\text{emit}}}^{a_{\text{obs}}}\frac{\mathrm{d}a}{a\dot{a}} = \frac{c}{R_0}\int_{(1+z)^{-1}}^{1}\frac{\mathrm{d}a}{a\dot{a}}$$

$$\chi = \frac{c}{R_0H_0}\int_{(1+z)^{-1}}^{1}\frac{\mathrm{d}a}{a^2}\frac{1}{\sqrt{\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} + \cdots + (1-\Omega_0)a^{-2}}}$$

The proper distance d_P, the angular distance d_A and the luminosity distance d_L could then be calculated using the value of $\chi(z)$,

$$d_P(t) = a(t)\chi$$
$$d_A = R_0S_k(\chi)(1+z)^{-1}$$
$$d_L = R_0S_k(\chi)(1+z)$$

2 Universe models

In this section we will solve the Friedmann equations for different universe models.

2.1 Empty universe (Milne)

The most simple model we may consider is an empty universe, with $\varepsilon = 0$. Equation (7) becomes:

$$H^2(t) = \frac{\dot{a}^2(t)}{a^2(t)} = -\frac{kc^2}{R_0^2a^2(t)} \qquad \textit{i.e.} \qquad \dot{a}^2(t) = -\frac{kc^2}{R_0^2} \tag{16}$$

This equation has two solutions. First, a static ($\dot{a} = 0$) and flat ($k = 0$) universe with no evolution.

But there is also another solution with:

$$k = -1 \qquad \dot{a}^2 = \frac{c^2}{R_0^2} \qquad \dot{a} = \pm\frac{c}{R_0} \tag{17}$$

For an expanding universe, the expansion is linear,

$$k = -1 \qquad a(t) = \frac{c}{R_0}t = \frac{t}{t_0} \qquad t_0 = \frac{1}{H_0} = \frac{R_0}{c} \tag{18}$$

In that universe,

$$a = \frac{1}{1+z} = \frac{t}{t_0} = H_0 t \tag{19}$$

This empty universe ("Milne universe") has no horizon:

$$d_H(t) = c\int_0^t \frac{\mathrm{d}t}{a(t)} = ct_0\int_0^t \frac{\mathrm{d}t}{t} \longrightarrow +\infty \tag{20}$$

2.2 Matter-dominated universes

In a universe containing only matter, $\Omega = \Omega_m$ and equation (7) becomes:

$$H^2(t) = \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{3c^2}\varepsilon_{m,0}\, a(t)^{-3} - \frac{kc^2}{R_0^2 a^2(t)} \tag{21}$$

This can also be written:

$$\frac{\dot{a}^2(t)}{H_0^2} = \frac{\Omega_0}{a(t)} + (1-\Omega_0) \tag{22}$$

2.2.1 Critical universe ($\Omega = \Omega_m = 1$, "Einstein-de Sitter")

For a flat universe ($k = 0$) containing only matter, the matter density is exactly equal to the critical density, and the Friedmann equation gives:

$$a(t) = \left[\frac{t}{t_0}\right]^{2/3} \qquad t_0 = \frac{2}{3H_0} \qquad \varepsilon_m(t) = \varepsilon_{m,0}\, a(t)^{-3} = \varepsilon_{m,0}\left[\frac{t}{t_0}\right]^{-2}$$

And the horizon distance at current time is:

$$d_H(t_0) = \frac{2c}{H_0}$$

See fig. 1.

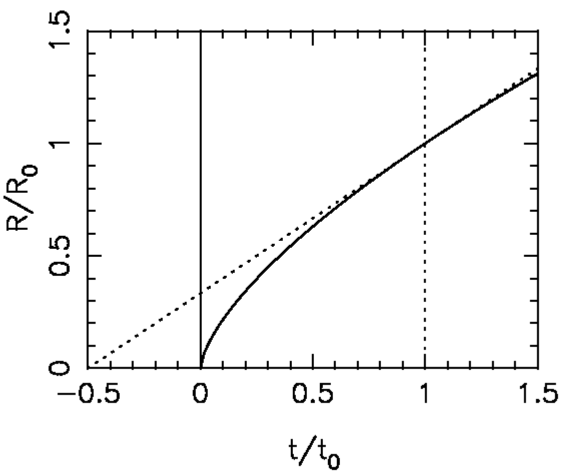

Figure 1: *Evolution of a flat matter dominated universe (critical universe, $\Omega = \Omega_m = 1$)*

2.2.2 Subcritical universe ($\Omega < 1$)

The matter density is below the critical density: $\Omega_m = \Omega < 1$. The universe will be open and negatively curved ($k = -1$), and the solution of equation (22) is:

$$a(\eta) = \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}(\cosh\eta - 1) = a_*(\cosh\eta - 1) \qquad \text{with} \quad a_* = \frac{R_*}{R_0} = \frac{4\pi G R_0^2 \varepsilon_0}{3c^4} = \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}$$
$$t(\eta) = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh\eta - \eta) = \frac{R_0 a_*}{c}(\sinh\eta - \eta)$$

where η goes from 0 to $+\infty$ (See fig. 3).

2.2.3 Supercritical universe ($\Omega > 1$)

When matter density is above the critical density: $\Omega_m = \Omega > 1$, the solution gives a closed universe with positive curvature ($k = +1$). Equation (22) then gives:

$$a(\eta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1 - \cos\eta) = \frac{a_{max}}{2}(1 - \cos\eta)$$
$$t(\eta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\eta - \sin\eta)$$

This is the parametric equation of a *cycloid*. The universe will expand and reach a maximum scale factor for $\eta = \pi$, at

$$a_{\text{max}} = a(\pi) = \frac{\Omega_0}{\Omega_0 - 1} = \frac{8\pi G R_0^2 \varepsilon_0}{3c^4} \qquad \text{at time} \quad t_{\text{max}} = t(\pi) = \frac{\pi}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{23}$$

At $\eta = 2\pi$, the *Big Crunch* will occurs:

$$a(2\pi) = 0 \qquad \text{at time} \quad t_{\text{crunch}} = t(2\pi) = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{24}$$

If each *Big Crunch* is followed by a new *Big Bang*, the universe may have a cyclic evolution (See fig 2).

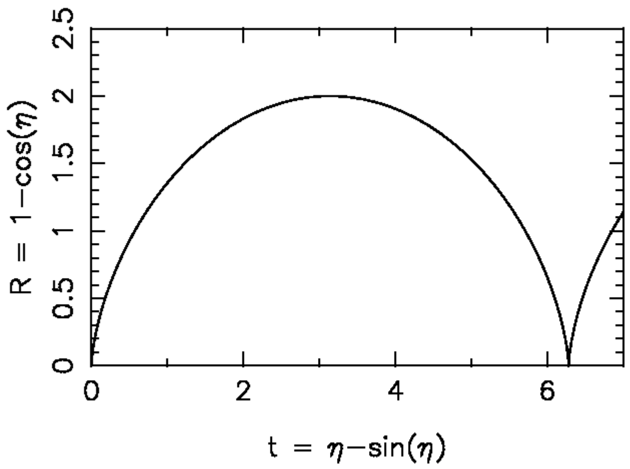

Figure 2: *Evolution of a cyclic universe ($\Omega = \Omega_m > 1$, $k = +1$)*

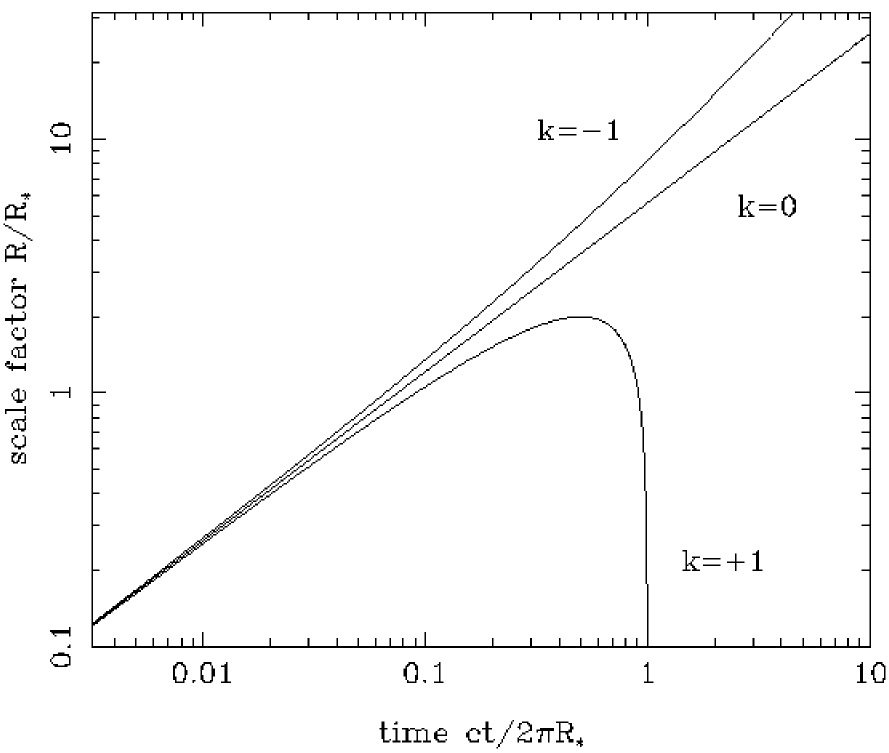

Figure 3: *Evolution of a universe dominated by matter, for a critical ($k = 0$), a subcritical ($k = -1$) and a supercritical ($k = +1$) universe.*

2.3 Vacuum-dominated universes (de Sitter)

For a universe containing only a cosmological constant ("vacuum" or "dark energy"), equation (7) becomes:

$$H^2(t) = \frac{\dot{a}^2}{a^2} = \frac{\Lambda c^2}{3} - \frac{kc^2}{R_0^2 a^2}$$
$$\dot{a}^2 = \frac{\Lambda c^2}{3}\, a^2 - \frac{kc^2}{R_0^2}$$

The solution is:

$$R(t) = R_0 a(t) = R_\Lambda \times \begin{cases} \cosh(t/t_\Lambda) & k = +1 \\ \frac{1}{2} e^{t/t_\Lambda} & k = 0 \\ \sinh(t/t_\Lambda) & k = -1 \end{cases}$$

Where

$$t_\Lambda = \frac{1}{c}\sqrt{\frac{3}{\Lambda}} \qquad R_\Lambda = R_0 a_\Lambda = c t_\Lambda = \sqrt{\frac{3}{\Lambda}}$$

These solutions give a universe which expands exponentially. In the $k = 0$ case, the expansion rate is constant:

$$H(t) = \frac{\dot{R}(t)}{R(t)} = \frac{\dot{a}(t)}{a(t)} = c\sqrt{\frac{\Lambda}{3}} = H_0$$

For $k = +1$, the scale factor has a minimum value $R_{\text{min}} = R_\Lambda$, which means that such a universe had no *Big Bang* (*Big Bounce* solutions).

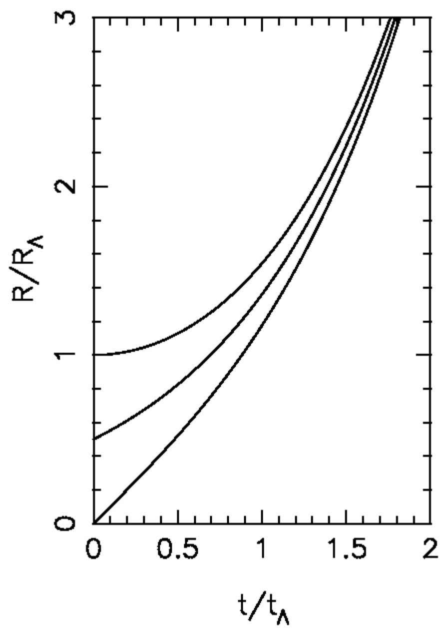

Figure 4: *Evolution of a universe dominated by a cosmological constant, for an open universe ($k = -1$, lower curve), a flat one ($k = 0$) and a closed universe ($k = +1$, upper curve).*

2.4 Flat universes

For a flat universe, $\Omega = 1$ and $k = 0$, equation (7) becomes:

$$H^(t) = \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{3c^2}\varepsilon(t)$$

If the universe contains only one fluid of density $\Omega_i = \Omega = 1$, which equation of state is $p = w\varepsilon$,

$$\dot{\varepsilon} + 3(1+w)\frac{\dot{a}}{a}\varepsilon = 0 \qquad \varepsilon(t) = \varepsilon_0\, a(t)^{-3(1+w)}$$

Then, for a flat universe containing a dominant fluid,

$$\dot{a}^2(t) = \frac{8\pi G}{3c^2}\varepsilon_0 a^{-(1+3w)}$$

The general solution is:

$$a(t) = \left[\frac{t}{t_0}\right]^{\frac{2}{3(1+w)}} \qquad t_0 = \frac{c}{1+w}\frac{1}{\sqrt{6\pi G\varepsilon}} \qquad H_0 = \frac{2}{3(1+w)}t_0^{-1} \qquad t_0 = \frac{2}{3(1+w)}H_0^{-1}$$

The energy density evolves with time as:

$$\varepsilon(t) = \varepsilon_0\, a^{-3(1+w)} = \varepsilon_0 \left[\frac{t}{t_0}\right]^{-2}$$

The horizon distance will be:

$$d_H(t) = c\frac{3(1+w)}{1+3w} t_0^{\frac{2}{3(1+w)}} t^{\frac{1+3w}{3(1+w)}} \qquad d_H(t_0) = ct_0\frac{3(1+w)}{1+3w} = \frac{c}{H_0}\frac{2}{1+3w}$$

If $w > -1/3$ there is a event horizon. On the opposite, if $w \leq -1/3$ there is no horizon: all space is causally connected and if the universe is transparent you can see all of it.

2.4.1 Matter dominated flat universe

As found previously, for a flat universe dominated by matter ($w_m \simeq 0$) we get:

$$a(t) = \left[\frac{t}{t_0}\right]^{2/3} \qquad t_0 = \frac{2}{3H_0} \qquad \varepsilon_m(t) = \varepsilon_{m,0}a(t)^{-3} = \varepsilon_{m,0}\left[\frac{t}{t_0}\right]^{-2}$$

And the horizon distance at current time is:

$$d_H(t_0) = \frac{2c}{H_0}$$

2.4.2 Radiation dominated flat universe

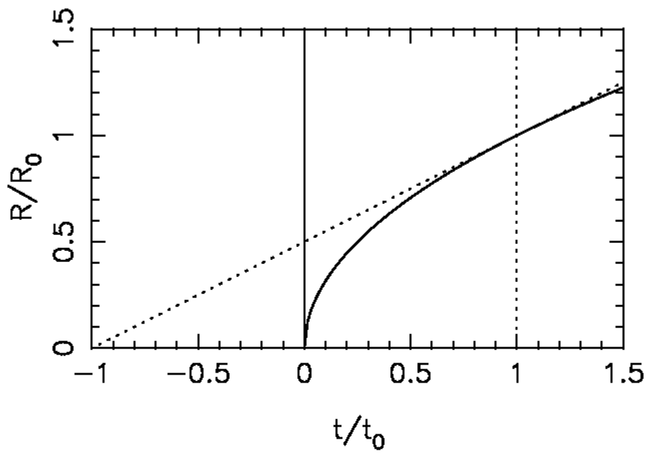

Figure 5: *Evolution of the scale factor* $a(t) = R(t)/R_0$ *for a radiation dominated universe.*

For a flat universe dominated by radiation ($w_r = 1/3$) we get:

$$a(t) = \left[\frac{t}{t_0}\right]^{1/2} \qquad t_0 = \frac{1}{2H_0} \qquad \varepsilon_r(t) = \varepsilon_{r,0}a(t)^{-4} = \varepsilon_{r,0}\left[\frac{t}{t_0}\right]^{-2}$$

And the horizon distance at current time is:

$$d_H(t_0) = \frac{c}{2H_0}$$

2.5 Matter and Λ

In a universe containing both matter and a cosmological constant Λ, $\Omega = \Omega_m + \Omega_\Lambda$ and the Friedmann equation may be written as

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \frac{1 - \Omega_{m,0} - \Omega_{\Lambda,0}}{a^2} + \Omega_{\Lambda,0}$$

The first and the last terms are positive, but the center one may be negative for a closed universe with $\Omega = \Omega_m + \Omega_\Lambda > 1$. Such a universe may exhibit very interesting behavior, depending of the values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ (See fig. 6 and 8). As $H^2(t)$ should be positive (otherwise it would be unphysical), Some combination of $(\Omega_{m,0}, \Omega_{\Lambda,0}, a(t))$ will not be allowed.

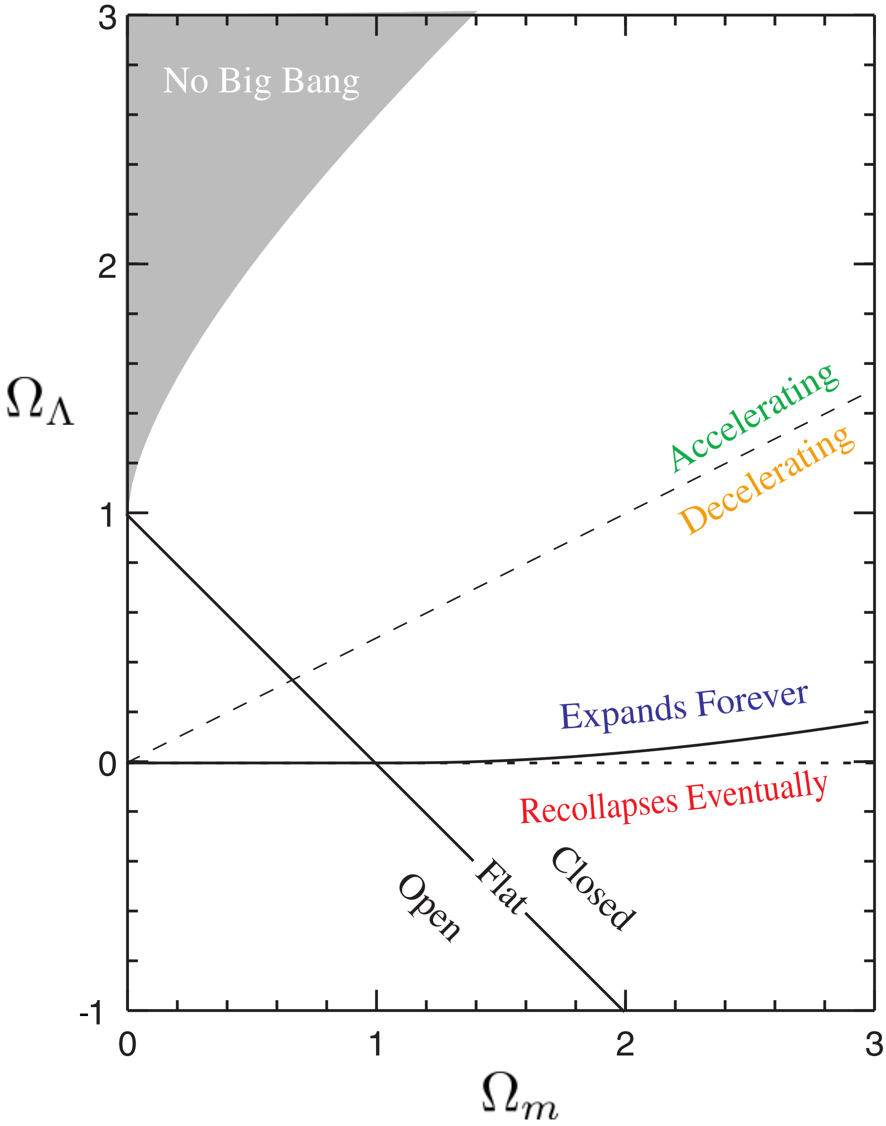

Figure 6: *Universe evolution depending on Ω_m and Ω_Λ, in the plane (Ω_m, Ω_Λ).*

2.5.1 Matter and Λ for a flat Universe

In that case, $\Omega = \Omega_m + \Omega_\Lambda = 1$ and the Friedmann equation becomes

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0})$$

The scale factor $a(t)$ is:

$$a(t) = \left[\sqrt{\frac{\Omega_{m,0}}{1-\Omega_{m,0}}} \sinh\left(\frac{3H_0\sqrt{1-\Omega_{m,0}}}{2}\, t\right)\right]^{2/3}$$

And the age of the Universe,

$$t_0 = \frac{2}{3H_0} \frac{1}{\sqrt{1-\Omega_{m,0}}} \operatorname{argsinh}\left(\sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}}\right)$$

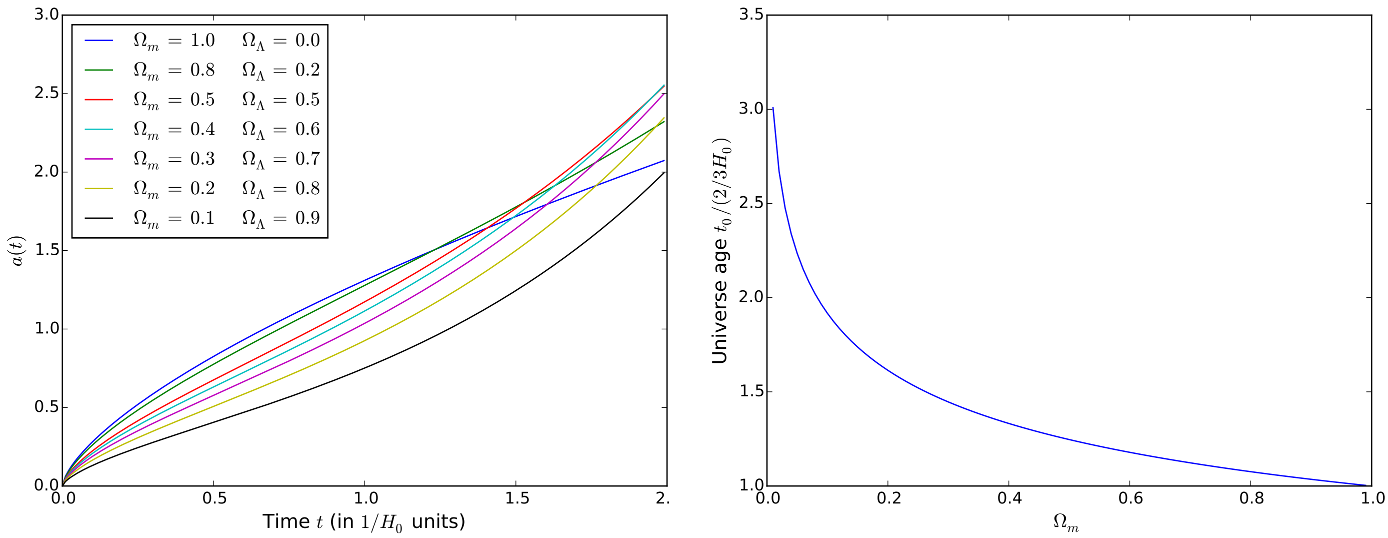

Figure 7: *Left: evolution of the scale factor $a(t)$ for a flat universe for different values of Ω_m and Ω_Λ. Right: age t_0 of a flat universe with $\Omega_m + \Omega_\Lambda = 1$ as a function of Ω_m.*

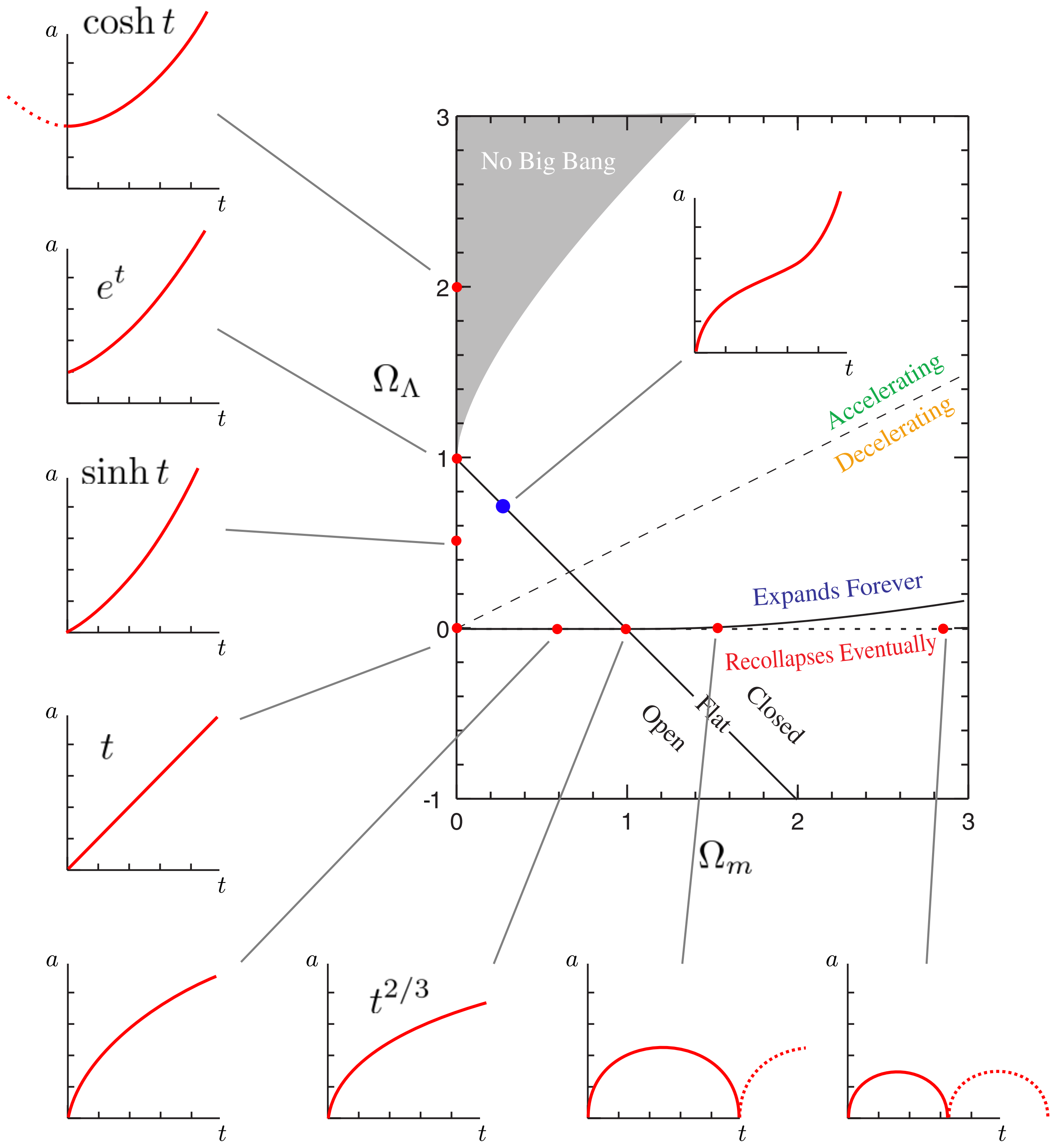

Figure 8: *Evolution of the scale factor $a(t)$ depending on Ω_m and Ω_Λ, in the plane (Ω_m, Ω_Λ).*

3 The Benchmark Model (*aka* Concordance Model, *aka* ΛCDM)

It seems that our Universe is flat ($k = 0$, $\Omega_0 = 1$). In early times, it was dominated by radiation,

$$a(t) \simeq \left(2\sqrt{\Omega_{r,0}}H_0 t\right)^{1/2} \qquad [a \ll a_{rm}]$$

Until the radiation-matter equality arount $a \simeq a_{rm} = \Omega_{r,0}/\Omega_{m,0} \simeq 2.8 \times 10^{-4}$. It was then dominated by matter,

$$a(t) \simeq \left(\frac{3}{2}\sqrt{\Omega_{m,0}}H_0 t\right)^{2/3} \qquad [a \gg a_{rm}]$$

In the next era, cosmological constant dominates. This era starts around $a_{m\Lambda} = (\Omega_{m,0}/\Omega_{\Lambda,0})^{1/3}$. Then the scale factor will increase exponentially,

$$a(t) \simeq a_{m\Lambda} \exp\left(\sqrt{\Omega_{\Lambda,0}}H_0 t\right) \qquad [a \gg a_{m\Lambda}]$$

Our Universe : recipe	
photons	$\Omega_{\gamma,0} \simeq 5.0 \times 10^{-5}$
neutrinos	$\Omega_{\nu,0} \simeq 3.4 \times 10^{-5}$
total radiation	$\Omega_{r,0} \simeq 8.4 \times 10^{-5}$
baryonic matter	$\Omega_{b,0} \simeq 0.04$
non-baryonic matter	$\Omega_{dm,0} \simeq 0.26$
total matter	$\Omega_{m,0} \simeq 0.3$
cosmological constant	$\Omega_{\Lambda,0} \simeq 0.7$

Epochs		
radiation-matter equality	$a_{rm} \simeq 2.8 \times 10^{-4}$	$t_{rm} \simeq 4.7 \times 10^4$ yr
matter-Λ equality	$a_{m\Lambda} \simeq 0.75$	$t_{m\Lambda} \simeq 9.8$ Gyr
now	$a_0 = a(t_0) = 1$	$t_0 \simeq 13.5$ Gyr

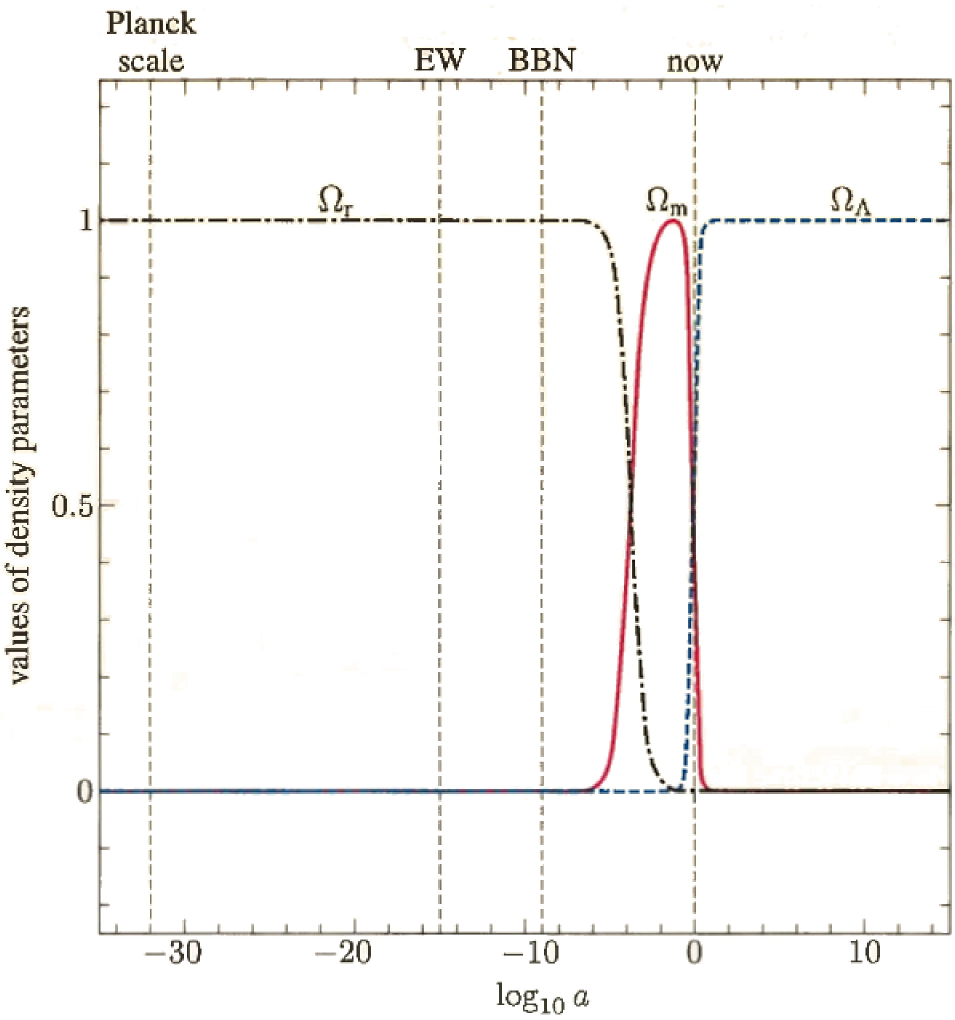

Figure 9: *Our Universe different eras.*