Academic Year 2015–2016
Master NPAC

MASTER NPAC– COSMOLOGY

EXAM

2016-02-02 9h30 – 12h30

Documents, books, computers and mobile phones are forbidden.
Pocket calculators are allowed.
Your answers may be written in English or in French, as you prefer.

1. Friedmann equations

1.1 — Give the Friedmann equations. We will use the convention in which the scaling factor $a(t)$ is dimensionless, and $a(t_0) = 1$ (t_0 is now). Explain the relation between a and z.

1.2 — Using the hypothesis that the expansion is an adiabatic process, show that, for a fluid of energy density $\varepsilon(t) = \rho(t)c^2$ and pressure p :

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + p) = 0$$

1.3 — For a given fluid, with an equation of state $p = w\varepsilon = w\rho c^2$, deduce the evolution of $\varepsilon(t)$ as a function of $a(t)$. How will the energy density evolve for *dust* (*i.e.* non-relativistic matter)? For *radiation* (*i.e.* relativistic matter)? For a cosmological constant Λ?

1.4 — What is the "critical density" $\varepsilon_c = \rho_c c^2$? Rewrite the Friedmann equation with the reduced densities $\Omega_X = \varepsilon_X/\varepsilon_c = \rho_X/\rho_c$.

1.5 — Give and justify the general expression for the current age t_0 of the universe for a mixing of matter, radiation and a cosmological constant (do not try to integrate at this point).

1.6 — Compute the age t_0 of a flat universe dominated by: (i) matter; (ii) radiation; (iii) a cosmological constant Λ.

1.7 — Compute the age t_0 of a flat universe dominated by matter and dark energy ($\Omega_0 = \Omega_{m,0} + \Omega_{\Lambda,0} = 1$). Give a literal expression (without integrating), and then using the following hint, the explicit expression of the age t_0.

Hints:

$$\int_A^B \frac{\mathrm{d}x}{\sqrt{x^2 + K/x}} = \left[\frac{2}{3}\ln\left(2\left[x^{3/2} + \sqrt{K + x^3}\right]\right)\right]_A^B \qquad \operatorname{argsinh} x = \ln\left(x + \sqrt{1 + x^2}\right)$$

Where $\ln x$ is the natural logarithm (*logarithme néperien*).

2. Variation of the redshift with time : $\mathrm{d}z/\mathrm{d}t_0$

Lets consider a source (a galaxy) at rest in comoving coordinates, at a redshift $z = a(t_0)/a(t_e) - 1$, where t_e is the time of light emission and t_0 is the observation time (now, $a(t_0) = 1$). We would like to determine how this redshift changes over the present time t_0, and estimate if this redshift variation could be detected, by measuring the redshift change of objects over a few years.

2.1 — Differentiate the expression of the redshift z with respect to the current time t_0 (be careful: photons received on Earth at $t_0 + \mathrm{d}t_0$ have been emitted at $t_e + \mathrm{d}t_e$: emission time t_e is a function of the reception time t_0). Show that

$$\frac{\mathrm{d}z}{\mathrm{d}t_0} = (1+z)H(t_0) - (1+z)H(t_e) \times \frac{\mathrm{d}t_e}{\mathrm{d}t_0}$$

2.2 — Recall the expression for the spacetime interval $\mathrm{d}s^2$ in the Friedman-Lemaître-Robertson-Walker (FLRW) metric. Express it with the *comoving* coordinates $\widetilde{\mathbf{x}} : x^\mu = (ct, \chi, \theta, \varphi)$, and also with the (ct, r, θ, φ) coordinates where $r = S_k(\chi)$,

$$S_k(\chi) = \begin{cases} \sin\chi & k = +1 \\ \chi & k = 0 \\ \sinh\chi & k = -1 \end{cases}$$

where χ is the (dimensionless) *comoving distance*.

2.3 — Express $\mathrm{d}s^2$ for a photon emitted by the galaxy (at $(t = t_e, \chi_{\mathrm{gal}}, \theta_{\mathrm{gal}}, \varphi_{\mathrm{gal}})$) and arriving later on earth (at $(t = t_0, \chi = 0, \theta_{\mathrm{gal}}, \varphi_{\mathrm{gal}})$). Deduce $\mathrm{d}\chi$ and the expression of the comoving distance of the galaxy χ_{gal}.

2.4 — Using the fact that the comoving distance χ_{gal} of the galaxy is constant, show that

$$\frac{\mathrm{d}t_e}{\mathrm{d}t_0} = \frac{a(t_e)}{a(t_0)}$$

(Hints: differentiate χ_{gal} relatively to t_0)

2.5 — Using the previous results, give $\mathrm{d}z/\mathrm{d}t_0$ as a function of z, t_0 and $H(z)$.

2.6 — For a flat universe with $\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$, evaluate (roughly) the redshift change Δz at $z = 1$ over 10 years (assume $H_0^{-1} \simeq 10^{10}$ years). Estimate the change in term of the apparent recession speed cz and its variation $c\Delta z$. Comment the result.

3. Matter perturbations in a flat matter-dominated universe

3.1 — Write the Friedmann equation for a flat, matter-dominated universe.

3.2 — Solve for $a(t)$, expliciting initial conditions.

3.3 — Derive the relation between the (average) matter density and t.

3.4 — Express $H(t)$.

To first order of perturbations, the evolution of a density perturbation in Newtonian dynamics follows:

$$\ddot{\delta} + 2H(t)\dot{\delta} = 4\pi G\bar{\rho}\delta$$

where $\delta \equiv \rho/\bar{\rho} - 1$, $\bar{\rho}$ denotes the average density and $\dot{\delta} = \mathrm{d}\delta/\mathrm{d}t$. The above equation assumes that pressure is negligible.

3.5 — Assuming that the universe is flat and matter-dominated, substitute H(t) and $\bar{\rho}(t)$ in the above equation.

3.6 — With the anzatz $\delta \propto t^n$, solve the differential equation for n.

3.7 — Comment the results.

4. The Cosmic Microwave Background

4.1 — Explain briefly the nature and the origin of the Cosmic Microwave Background (CMB).

4.2 — Describe briefly how the curve of figure 1 is derived from a measured CMB temperature map on the celestial sphere $T(\theta, \phi)$. Derive from this curve an order of magnitude of the variance or (r.m.s) of $T(\theta, \phi)$.

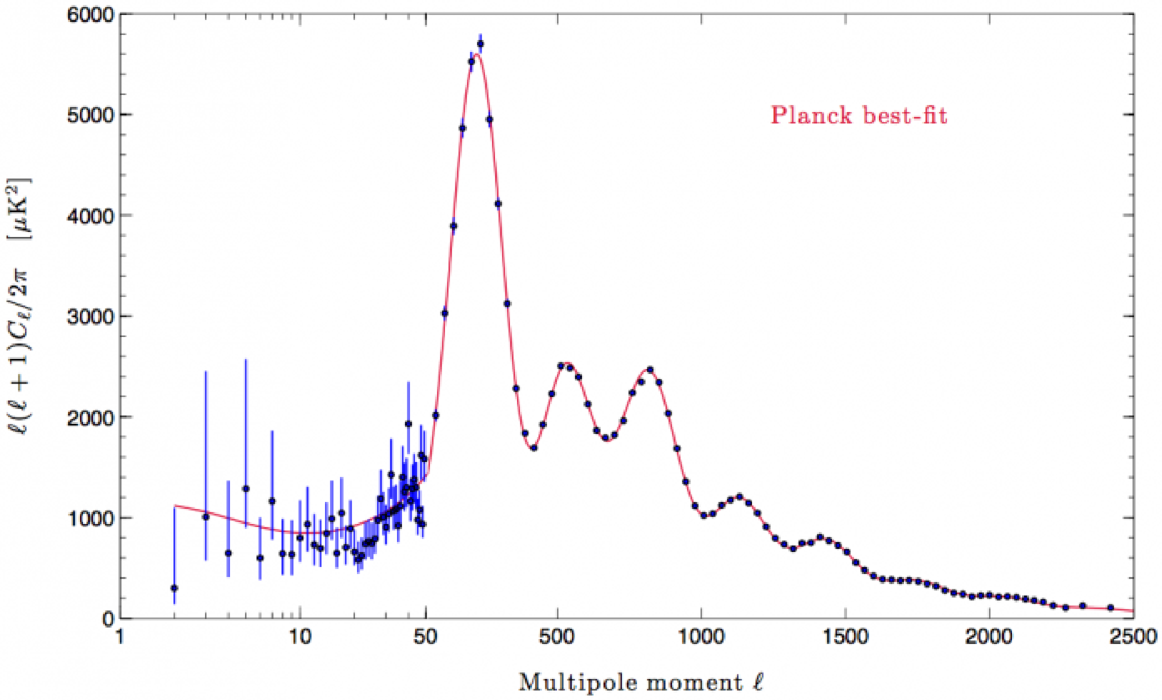

Figure 1: *CMB spectrum. Logarithmic x-scale up to $\ell = 50$; linear at higher ℓ; all points with error bars. The red line is the Planck best-fit primordial power spectrum (cf. Planck+WP+highL in Table 5 of Planck-2013-XVI, A&A 571, A16 (2014)). From the Planck Collaboration.*

4.3 — Give a rough relation between ℓ values and real angles on the sky. Where are the large angular scales? Where are the small angular scales?

4.4 — One can depict this curve as

1. a plateau at low ℓ values;
2. a series of peaks at higher ℓ;

3. a global enveloppe damping the high ℓ.

Give a short description of the physics involved in each of these features, and if applicable the relevant parameters.

5. Inflation

5.1 — During a period of inflation, what is the sign of $\ddot{a}$? Does a cosmological constant give rise to inflation, and if so, is it a good model for an inflationary phase early in the history of the universe? (Explain your answer.)

5.2 — How does inflation solve the horizon problem? Draw a plot of the comoving scale factor as a fun ction of the scale factor to explain your answer. How long should inflation last approximately?

5.3 — Show that the acceleration equation

$$\dot{H} = -4\pi G\dot{\phi}^2 \tag{1}$$

is a consequence of the Friedman equation as well as the continuity equation, respectively

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \qquad \ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \tag{2}$$

What is the quantity $V(\phi)$ appearing in these equations?

The slow roll parameters were defined in lectures as:

$$\epsilon_V \equiv \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{V_\phi}{V}\right)^2 \qquad \eta_V \equiv M_{\mathrm{Pl}}^2\left(\frac{V_{\phi\phi}}{V}\right) \tag{3}$$

where $V_\phi = \mathrm{d}V/\mathrm{d}\phi$, $V_{\phi\phi} = \mathrm{d}^2V/\mathrm{d}\phi^2$ and

$$M_{\mathrm{Pl}} = \frac{1}{\sqrt{8\pi G}}.$$

For simplicity in the following we work in units in which $M_{\mathrm{Pl}} = 1$. Also, let N denote the *number of e-folds before the end of inflation*:

$$N(t) = \ln\frac{a_{\mathrm{end}}}{a(t)}$$

where a_{end} is the scale factor at the end of inflation.

5.4 — Why are ϵ_V and η_V called "slow roll" parameters? Which approximations must be made to obtain them from equations (1)-(2)? Explain why inflation requires $\epsilon_V \ll 1$ and $\eta_V \ll 1$.

5.5 — For the remainder of this exercise we consider the potential

$$V(\phi) = \alpha\phi^2 \tag{4}$$

where α is a constant. What are the dimensions of α?

5.6 — Calculate the slow-roll parameters, and determine the value of ϕ at the end of inflation.

5.7 — Show that, in the slow-roll approximation (that is, to first order in slow-roll parameters),

$$N \simeq \int_{\phi_{\text{end}}}^{\phi} \frac{d\phi}{\sqrt{2\epsilon_V}}.$$

Calculate the number N of e-folds between t_i where the field takes the value ϕ_i, and the end of inflation.

5.8 — Determine $\phi(t)$ and $a(t)$ in the slow-roll approximation.